

Framed BPS States, Moduli Dynamics, and Wall-Crossing

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Abstract

We formulate supersymmetric low energy dynamics for BPS dyons in strongly-coupled $N = 2$ Seiberg-Witten theories, and derive wall-crossing formulae thereof. For BPS states made up of a heavy core state and n probe (halo) dyons around it, we derive a reliable supersymmetric moduli dynamics with $3n$ bosonic coordinates and $4n$ fermionic superpartners. Attractive interactions are captured via a set of supersymmetric potential terms, whose detail depends only on the charges and the special Kähler data of the underlying $N = 2$ theories. The small parameters that control the approximation are not electric couplings but the mass ratio between the core and the probe, as well as the distance to the marginal stability wall where the central charges of the probe and of the core align. Quantizing the dynamics, we construct BPS bound states and derive the primitive and the semi-primitive wall-crossing formulae from the first principle. We speculate on applications to line operators and Darboux coordinates, and also about extension to supergravity setting.

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1 Introduction

The wall-crossing in supersymmetric theories refers to the phenomenon where certain one-particle BPS states [1, 2] disappear from the spectrum as the vacuum moduli or parameters are changed continuously. The naive stability argument of BPS states relies on the short-multiplet structure due to partially preserved supersymmetry, but this is really applicable only when we consider dynamical processes in a given vacuum. When we change vacuum or parameters, even continuously, the state itself can disappear from the spectrum altogether at which point the supermultiplet structure of the state becomes a moot issue.

Although the wall-crossing had an early precursor in the context of supersymmetric kinks in two-dimensional $N = (2, 2)$ theories [3, 4], it was in the context of $N = 2$ supersymmetric theories in four dimensions, such as Seiberg-Witten theory [5, 6] and Calabi-Yau compactification of type II string theories, that the phenomenon came under wide scrutiny. The co-dimension one surface across which a BPS state disappear is called the marginal stability wall (MSW), and their presence renders the problem of finding BPS spectrum extremely complicated. For the simplest of Seiberg-Witten theories, monodromy properties [5] alone can determine the spectrum [7] but this is more an exception than a rule.

Despite such early discoveries, the space-time picture of exactly what happens to the state upon the wall-crossing remained unclear until it was uncovered in the context of $1/4$ BPS dyons in $N = 4$ super Yang-Mills theory, which preserve four supersymmetries just as $1/2$ BPS objects of $N = 2$ theories do. It was found in Ref. [8] that such BPS dyons must be, typically, thought of as a loose bound states of more than one dyonic centers with mutually non-local charges. The distances between such centers are not free but determined by the vacuum moduli u_i 's

$$R_{AB} = R_{AB}(u_i; \{q_C^i, p_C^i\}) , \quad (1.1)$$

with (q_A^i, p_A^i) being the charges of the A -th center. In particular, some of R_{AB} was shown to diverge as a MSW is approached; this happens simply because scalar forces and electromagnetic forces do not cancel each other between dyons of mutually non-local charges and the equilibrium distance is determined by a detailed balance of classical forces: state by state, the wall-crossing has a very mundane and classical explanation.

This finding is immediately applicable to weakly coupled $N = 2$ theories as well, because $N = 2$ theory BPS solitons can be classically embedded to a $N = 4$ theory. There will be differences at quantum level because the supermultiplet structures (and flavor structures) are different, but the above space-time picture of wall-crossing is essentially classical and quite robust.

For both $N = 2$ and $N = 4$ theories, this multi-center nature and the subsequent wall-crossing were soon elevated to the semiclassical level [9, 10, 11]. The quantum low energy dynamics of magnetic monopoles was derived rigorously from the super Yang-Mills theories in question [12], and dyons were constructed as quantum bound states of monopoles with certain conjugate momenta turned on [13, 10]. What used to be the classical orbit size is now represented by the quantum bound state size, and is still determined by the vacuum moduli and charges as in (1.1). The size of the bound state is again divergent as a wall of marginal stability is approached, across which the state no longer exists as quantum and BPS one-particle state.

In such supersymmetric low energy dynamics of solitons, precise state counting is a simple matter of finding bound state wavefunctions or computing index of certain Dirac operators on the moduli space. For instance, the bound state of a pair of dyons of charge $\gamma_1 + \gamma_2$ has been constructed and counted when the total magnetic charge is a dual root. The degeneracy on one side of a wall [10]^{#1} can be written as

$$|\Omega(\gamma_1 + \gamma_2)| = 2|\langle\gamma_1, \gamma_2\rangle| , \quad (1.2)$$

where we introduced $\Omega(\gamma)$, the second helicity trace for the supermultiplet of charge γ , as

$$\Omega = -\frac{1}{2}\text{tr}((2J_3)^2(-1)^{2J_3}) . \quad (1.3)$$

A simplest generalization of this is a chain of dyons with nearest neighbor interactions, namely $\langle\gamma_A, \gamma_B\rangle \neq 0$ if and only if $|A - B| = 1$. Whenever such a state exists as a quantum BPS state, the degeneracy takes a simple form [14],

$$|\Omega(\gamma_1 + \gamma_2 + \gamma_3 + \cdots)| = \left| \prod_A 2\langle\gamma_A, \gamma_{A+1}\rangle \right| . \quad (1.4)$$

They were shown to exist only when each and every one of $\langle\gamma_A, \gamma_{A+1}\rangle$ obeys certain inequalities defined by the vacuum moduli, which amounts to being on the “right” side of several MSW’s, basically one for each interacting pair. The formula clearly suggests that such states can be constructed iteratively by attaching one kind of dyons at a time, already hinting at a simple universal wall-crossing formula.

The next breakthrough came from $N = 2$ supergravity analysis by Denef who also found the multi-centered nature of BPS black holes and the subsequent wall-crossing in the context of attractor flow solutions [15, 16]. The approach gave a universal and explicit constraints for the relative positions of charge centers, say, charge γ_A at \vec{x}_A ,

$$\sum_{B \neq A} \frac{\langle\gamma_B, \gamma_A\rangle}{|\vec{x}_B - \vec{x}_A|} = \text{Im}[\zeta_T^{-1} Z(\gamma_A)] . \quad (1.5)$$

^{#1}In this note, we take the convention that Schwinger products take values in $\mathbf{Z}/2$.

where $Z(\gamma_A)$ is the central charge of γ_A and ζ_T is the phase factor of the total central charge $Z_T = \sum_A Z(\gamma_A)$. In supergravity, this simplifies and supersedes the field theory results which we abstractly noted as (1.1).

The wall-crossing for supergravity black hole solutions is again due to a divergent distance between the charge centers, which is dictated by long distance classical physics, just as in the field theory soliton picture of BPS dyons: the sign of the left hand side of (1.5) is independent of vacuum moduli while that of the right hand side can flip the sign as we change vacuum. Clearly at some point where the right hand side approaches zero from the positive side, one of the distances has to diverge, beyond which the solution can no longer exist. This is most useful since the sizes of the states can be found without detailed construction. However, there is no information on how a given charge state is split into what charge centers, unlike the explicit constructions of multi-center solution/quantum states in the field theory story.

Although supergravity solutions themselves were not amenable to explicit and precise quantum counting, Denef further went on to conjecture general two-body wall-crossing formula that extends the above field theory result to arbitrary (magnetic) charges [17]. With spin content taken into account, the formula reads,

$$\Omega(\gamma_1 + \gamma_2) = -(-1)^{2|\langle \gamma_1, \gamma_2 \rangle|} 2|\langle \gamma_1, \gamma_2 \rangle| \Omega(\gamma_1) \Omega(\gamma_2) , \quad (1.6)$$

which was later extended by Denef and Moore to the semi-primitive cases [18], captured in a generating function,

$$\sum_{n=0} \Omega(\gamma_1 + n\gamma_2) q^n = \Omega(\gamma_1) \prod_{k=1} \left[1 - (-1)^{2k\langle \gamma_2, \gamma_1 \rangle} q^k \right]^{2k|\langle \gamma_2, \gamma_1 \rangle| \Omega(k\gamma_2)} , \quad (1.7)$$

counting the BPS states of charges $\gamma_1 + n\gamma_2$ in terms of degeneracies of states with charges γ_1 and $n\gamma_2$. These spurred much activities toward general solutions to the wall-crossing problem, and was integrated recently into more general Kontsevich-Soibelman's wall-crossing formalism [19].

Despite evidences that support the semi-primitive wall-crossing formulae of Denef-Moore (which in turn support Kontsevich-Soibelman formalism), it has been rigorously tested only in specific cases. The most systematic example of this is the $N = 2$ weak coupling analysis that preceded the conjecture, but the limitation of weak coupling limit casts some shadows on its general usefulness. It would be very useful if we can find a similarly systematic method of constructing and counting BPS bound states, and apply to diverse BPS objects, such as those dyons that appear in the generic strongly coupling region of Seiberg-Witten theory. In this paper, we wish to initiate a new framework that can count and construct BPS states, without referring either to specific subset of charges or to weak electric coupling, but applicable to a large class of $N = 2$ theories and BPS states thereof.

One common lesson from earlier studies of multi-centered BPS states is that non-Abelian completion of the state at charge centers is not essential for understanding wall-crossing, since the latter is essentially a long distance phenomenon from the spacetime viewpoint. The supergravity solutions are all Abelian while, for solitons, it is the long range Coulomb-type interactions that determined the multi-center nature of the state. Related is the notion of the “framed” BPS state [20]. The main idea there was to treat one or more component dyons as an external object, and the remainders as dynamical object around such a background. This way, one can treat the former as the background, in which the latter moves around and sometime becomes supersymmetrically bound to the core state. This split of the state into two parts can simplify the state construction and counting substantially.

Inspired by these ideas, we wish to consider dyons moving around purely Abelian dyonic background. In effect, we will split the state in question into the heavy “core state” of total charge γ_c and light “halo” or “probe” of charge γ_h . For our purpose, it is the ratio of the two masses that matters, so this can be for instance achieved by approaching a singular point where the probe dyon becomes massless. The low energy dynamics of the probe dyon is quite natural thing to do there since, precisely at such a singular point, the probe dyon would be the lightest particle among charged states. However, there are other circumstances where one part become relatively light compared to the other, and our framework will apply.

Another useful fact is that, as far as wall-crossing behavior goes, we only need information near the relevant MSW’s, away from which the BPS spectrum is continuous. This allows another small quantity to play with, by taking vacuum very near a marginal stability wall. As we will see later, the distance to the MSW plays a role very similar to the weak electric coupling in that it controls the nonrelativistic approximation. In the end, we find that the dynamics between the core and the probe reduces to massive supersymmetric quantum mechanics with two kinds of potentials.

These two lead us to a new model of low energy dynamics for dyons in the strongly coupled region of $N = 2$ field theory. Although similar in spirit to the old moduli dynamics of solitons, an essential difference here is that the small electric coupling constant is no longer needed; this is what allows us to apply the technique to much wider class of BPS states than previously possible.^{#2} The quantum mechanics has four supercharges, as required by the BPS condition, but comes with only $3n$ bosonic coordinates, three for each probe dyon, and $4n$ fermionic coordinates. Compared to the conventional moduli dynamics of weakly coupled regime, we are missing one angular collective coordinate for each dyon. This has something to do with the fact that we start with dyons, rather than monopoles, as basic building blocks.

^{#2}In fact, it should be possible to extend this framework to include gravity and discuss quantum bound states of charged BPS black holes.

With this new low energy dynamics in place, we can compute how many BPS bound states of the core and the probe dyons can form, and under what condition. At the end of day we derive, via a first-principle computation, the semi-primitive wall-crossing formula with $\gamma_1 = \gamma_c$ and $\gamma_2 = \gamma_h$. In this note there is in fact no restriction on γ_c , as far as such a state actually exist and all of its component dyon centers can be made heavy. Thus we in effect are computing $\Omega(\gamma_c + n\gamma_h)$ with the only restriction that the dyon γ_h is primitive and become massless somewhere in the vacuum moduli space. We wish to emphasize that, alternatively, we may think of the theory as a setup for finding framed BPS state with line operator of charge γ_c and halos γ_h [20].

This paper is organized as follows. In Section 2, we write down the long-distance Abelian form of the core state in terms of the central charge function, while the probe dyons are treated as quantized solitons in that background. As a result we find a bosonic low energy Lagrangian of the probe dyons purely in terms of quantities that can be constructed out of the central charge functions. This reproduces some of general results, such as distances between two charge-centers, obtained from supergravity attractor flow analysis, even though we are dealing with field theory states.

Section 3 discusses how one can construct a $\mathcal{N} = 4$ supersymmetric Lagrangian with $3n$ bosonic coordinates and $4n$ fermionic coordinates, by extending previous studies by Coles and Papadopoulos [21] and also by [22]. These previous works constructed massless supersymmetric theories of similar kind, which is, however, missing the crucial elements of potentials. Without the latter, the bound states we are interested in cannot form at all. We construct in particular massive theories in which degrees of freedom are cataloged by $SO(4)_R = SU(2)_L \times SU(2)_R$ algebra with bosons in $(\mathbf{3}, \mathbf{1})$ (thus, the first $SU(2)_L$ also serves as a rotation group) and fermions in $(\mathbf{2}, \mathbf{2})$ representations. The four supercharges are also in $(\mathbf{2}, \mathbf{2})$. The Lagrangian has $SU(2)_R$ symmetry manifest while $SU(2)_L$ can be explicitly broken by the background.

Section 4 shows how the general discussion of section 3 makes contact with the probe dyon dynamics of section 2 under the assumption the vacuum moduli of the underlying $N = 2$ theory is very near the MSW. The latter assumption controls the energy scale of the potential energy, and allows a nonrelativistic approximation possible. We then quantize the resulting dynamics and derive the bound states for $\gamma_c + \gamma_h$, and again shows how the bound state size diverges as one approach MSW and how the bound state is impossible on the other side of MSW.

Section 5 elevates this to a primitive wall-crossing formula, and extends further to the cases of $\gamma_c + n\gamma_h$ by invoking spin-statistics theorem. This derives, in particular, the semi-primitive wall-crossing formula from a first principle computation.

We then conclude in Section 6 with summary and other comments especially on

how one can make use of this formalism to compute the line operator expectation values and how one can extend the formalism to the supergravity setting. Some computational details are summarized in Appendices.

2 Classical Dynamics of Probe Dyons

In this section, we construct the semiclassical form of the core state, entirely in terms of the central charge function, and describe energetics and dynamics of a probe dyon in the core state background. This leads us to a bosonic Lagrangian of the probe dyon, which will be supersymmetrized and quantized in the later section.

Although the exercise here applies to any core state one can imagine, as long as there solve the relevant semi-classical BPS equation of the effective Abelian theory, we are eventually interested in core states that actually exist as quantum BPS states. It is known that the former does not always imply the latter [23, 10]. Alternatively, for the framed BPS states, the core state should correspond to a supersymmetric line operator. Either way, we are interested in case where the supersymmetric lift of this probe bosonic dynamics would make sense in the context of the underlying four-dimensional theory.

2.1 Semiclassical Core State

We start by recalling semiclassical properties of $N = 2$ dyons when expressed in terms of the low energy theory of Seiberg and Witten. Traditionally the smooth solitons are possible only when we include the entire non-Abelian origin, but this is practical only in the weakly coupled limit.

To avoid such restrictions, a more convenient starting point is to write the BPS equation in the Abelian low energy description of Seiberg and Witten. This approach was investigated previously [24, 25] with emphasis on split flow picture of the classical soliton and gave an interesting parallel to the string web picture [26] of $N = 4$ $1/4$ BPS dyons. These solutions are invariably singular at the charge centers, since there is no non-Abelian mechanism to stop the Coulomb-like divergence at origin, which was controlled ad hoc by introducing UV cutoffs.

For our purpose, however, this divergence is of little consequence, essentially because we will be using this solution as background. As long as we can ascertain existence of quantum state of such a charge and as long as we put correct boundary condition at such singular points, forcing the probe dyon wavefunction to vanish there fast enough, there would be no physical problem associated with it. It is entirely anal-

ogous to the Hydrogen atom problem of undergraduate quantum mechanics, where finite and trustworthy bound states are obtained even though the Hamiltonian is naively singular at origin.

Using the SUSY transformation rule for gaugino along the particular direction parameterized by a phase factor ζ , one can obtain the BPS equations

$$\vec{\mathcal{F}}_i - i\zeta^{-1}\vec{\nabla}\phi_i = 0 , \quad (2.1)$$

where i labels the unbroken $U(1)$ gauge groups, and $\vec{\mathcal{F}}$ denotes the complexified field strength 3-vector $\vec{B} + i\vec{E}$. See appendix A for details. There is also an electric version of this equation

$$\vec{\mathcal{F}}_D^i - i\zeta^{-1}\vec{\nabla}\phi_D^i = 0 , \quad (2.2)$$

with $\vec{\mathcal{F}}_D^i \equiv \tau^{ij}\vec{\mathcal{F}}_j$ and

$$\tau^{ij} = \frac{\partial^2}{\partial\phi_i\partial\phi_j} F_{\text{SW}}(\phi) , \quad (2.3)$$

where $F_{\text{SW}}(\phi)$ is the Seiberg-Witten prepotential of the given theory. Since it is $\text{Re}\mathcal{F}_D$ that enters the Gauss constraint, the field strengths are such that [24]

$$\text{Re} \int_{S_\infty^2} \mathcal{F}_i = 4\pi P^i, \quad \text{Re} \int_{S_\infty^2} \mathcal{F}_D^i = -4\pi Q^i, \quad (2.4)$$

with the total magnetic charges P^i and the total electric charges Q^i

In particular imagine a semiclassical core state of charges $\gamma_c = (P^i, Q^i) = \sum_A \gamma_{c,A}$, with $\gamma_A = (P_A^i, Q_A^i)$, distributed into several dyonic cores at \vec{x}^A , and the field strength takes the following asymptotic forms,

$$\begin{aligned} \text{Re} \vec{\mathcal{F}}^i &= \sum_A \frac{P_A^i(\vec{x} - \vec{x}_A)}{|\vec{x} - \vec{x}_A|^3} = \vec{\nabla} \left(- \sum_A \frac{P_A^i}{|\vec{x} - \vec{x}_A|} \right) , \\ \text{Re} \vec{\mathcal{F}}_D^i &= - \sum_A \frac{Q_A^i(\vec{x} - \vec{x}_A)}{|\vec{x} - \vec{x}_A|^3} = \vec{\nabla} \left(\sum_A \frac{Q_A^i}{|\vec{x} - \vec{x}_A|} \right) . \end{aligned} \quad (2.5)$$

One can show that ζ can be identified as the phase factor of central charge Z_{core} of this core state^{#3}

$$Z_{\text{core}} = |Z_{\text{core}}|\zeta = Q^i\phi_i(\infty) + P_i\phi_D^i(\infty) . \quad (2.6)$$

This semiclassical description is, strictly speaking, valid away from $\vec{x} = \vec{x}_A$'s.

^{#3}See Appendix A.

Note that the positions, \vec{x}_A 's, of the centers would be restricted by an analog of (1.5). Precise positions of these centers is, however, immaterial for counting BPS bound states, as long as the relevant core state actually exists as quantum and BPS bound state. This happens because one ends up computing supersymmetric indices, which are robust under small deformations of the supercharges. More important is how the core electromagnetic charge is distributed into such centers. See section 4 for related discussions.

2.2 Probe Dyons and Electromagnetic Forces

Let us now introduce a probe particle of charge $\gamma_h = (p_i, q_i)$, in a background created by such a core state. It will be considered as a probe particle in the external electromagnetic field by the massive core state. Using the equations (2.1, 2.2), one obtains

$$q \cdot \vec{\mathcal{F}} + p \cdot \vec{\mathcal{F}}_D = i\zeta^{-1} \vec{\nabla} \mathcal{Z}_h, \quad (2.7)$$

where $\mathcal{Z}_h = q \cdot \phi + p \cdot \phi_D$ is now understood as position-dependent. We introduced the notation \mathcal{Z} to emphasize that this quantity is position-dependent. The usual central charge Z is related to it as $Z = \mathcal{Z}(\infty)$.

The real and imaginary part of the relation will give us hints how to construct the low-energy Lagrangian of probe dyon in the background of core particle. The real part can be succinctly written as

$$\vec{\nabla} V_{\text{Coulomb}} = -\vec{\nabla} \text{Re} \left[\zeta^{-1} \mathcal{Z}_h \right], \quad (2.8)$$

where

$$\begin{aligned} V_{\text{Coulomb}} &= \text{Re}(\tau)_{ij} \sum_A \frac{p^i P_A^j}{|\vec{x} - \vec{x}_A|} \\ &+ (\text{Re}(\tau))_{ij}^{-1} \sum_A \frac{(q_i + \text{Im}(\tau)_{ij} p^j) (Q_{j,A} + \text{Im}(\tau)_{ij} P_A^j)}{|\vec{x} - \vec{x}_A|} \end{aligned} \quad (2.9)$$

is nothing but the Coulomb potential energy felt by the probe dyon due to the core state. The real part of this equation is even simpler

$$\vec{\nabla} \left(\sum_A \frac{Q_A \cdot p - P_A \cdot q}{|\vec{x} - \vec{x}_A|} \right) = -\vec{\nabla} \text{Im} \left[\zeta^{-1} \mathcal{Z}_h \right], \quad (2.10)$$

or equivalently

$$\vec{\nabla} \left(\sum_A \frac{\langle \gamma_{c,A}, \gamma_h \rangle}{|\vec{x} - \vec{x}_A|} \right) = -\vec{\nabla} \text{Im} \left[\zeta^{-1} \mathcal{Z}_h \right]. \quad (2.11)$$

which, as we will presently see, encodes the Lorentz force on the probe dyon.^{#4}

We first discuss the invariant expression of the minimal coupling under the Montonen-Olive duality. Recall that, from the BPS equations, one can conclude that $(\vec{\mathcal{F}}, \vec{\mathcal{F}}_D)$ transform under the duality transformation as vector representation like (ϕ, ϕ_D) . For example, let us consider the S-duality transformation of $(\vec{\mathcal{F}}, \vec{\mathcal{F}}_D)$

$$\vec{B} \rightarrow -\text{Im}(\tau)\vec{E} + \text{Re}(\tau)\vec{B}, \quad \vec{E} \rightarrow \text{Im}(\tau)\vec{B} + \text{Re}(\tau)\vec{E}. \quad (2.12)$$

Then, one can easily show that, under the S-duality transformation,

$$q \rightarrow p, \quad p \rightarrow -q, \quad (2.13)$$

where we used, for the last transformation, the fact that $\tau \rightarrow -\tau^{-1}$.

When the probe dyon moves (slowly) under the electromagnetic field of core particle, the minimal coupling terms therefore become [27, 28]

$$\mathcal{L}_{\text{int}} = q^i \vec{v} \cdot \vec{A}^i + p^i \vec{v} \cdot \vec{\tilde{A}}^i + q^i A_0^i + p^i \tilde{A}_0^i, \quad (2.14)$$

which is the duality invariant expression. Here A_μ and \tilde{A}_μ are defined as

$$\begin{aligned} \text{Re}\vec{\mathcal{F}} &= \vec{\nabla} \times \vec{A}, & \text{Re}\vec{\mathcal{F}}_D &= \vec{\nabla} \times \vec{\tilde{A}}, \\ \text{Im}\vec{\mathcal{F}} &= \vec{\nabla} \cdot A_0, & \text{Im}\vec{\mathcal{F}}_D &= \vec{\nabla} \cdot \tilde{A}_0. \end{aligned} \quad (2.15)$$

Using the BPS equation (2.7), the interaction terms can be managed into a rather simpler form

$$\mathcal{L}_{\text{int}} = -\vec{v} \cdot \vec{\mathcal{W}} + \text{Re}[\zeta^{-1} \mathcal{Z}_h(x)] - \text{Re}[\zeta^{-1} \mathcal{Z}_h(\infty)], \quad (2.16)$$

where the vector \vec{w} satisfies the relation below

$$\vec{\nabla} \times \vec{\mathcal{W}} = \vec{\nabla} \text{Im}[\zeta^{-1} \mathcal{Z}_h(x)]. \quad (2.17)$$

Note that, in (2.16), $\text{Re}[\zeta^{-1} \mathcal{Z}_h(\infty)] = \text{Re}[\zeta^{-1} Z_h]$ represents the lowest possible energy the probe dyon can attain.

^{#4}The normalization of charges and the sign convention for Schwinger product differs from that of Refs. [15, 18]

$$\langle \gamma, \gamma' \rangle = \frac{1}{2} \langle \gamma', \gamma \rangle_{\text{Denef-Moore}}$$

Related is the fact that our ζ is $-\zeta_{\text{Denef-Moore}}$.

2.3 Massive Moduli Dynamics of Probe Dyons

Finally we come to the effect of the long range scalar field on the dyon. The low-energy Lagrangian of probe dyon γ_h moving in the background of core particle γ_c can take the following form

$$\mathcal{L}^{\text{bosonic}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} , \quad (2.18)$$

where the kinetic term must be [27, 28]

$$\mathcal{L}_{\text{kin}} = -|\mathcal{Z}_h(x)|\sqrt{1-v^2} \simeq -|\mathcal{Z}_h(x)| + \frac{1}{2}|\mathcal{Z}_h(x)|\vec{v}^2 + \mathcal{O}(v^4) \quad (2.19)$$

with $\mathcal{Z}_h(x) = q \cdot \phi + p \cdot \phi_D$, since $|\mathcal{Z}_h(x)|$ is the effective inertia of the probe dyon. Adding all these together, we find the classical Lagrangian,

$$\mathcal{L}^{\text{bosonic}} = \frac{1}{2}|\mathcal{Z}_h(x)|\vec{v}^2 - |\mathcal{Z}_h(x)| + \text{Re}(\zeta^{-1}\mathcal{Z}_h(x)) - \text{Re}[\zeta^{-1}\mathcal{Z}_h(\infty)] - \vec{v} \cdot \vec{\mathcal{W}} \quad (2.20)$$

with $\vec{\nabla} \times \vec{\mathcal{W}} = \vec{\nabla} \text{Im}(\zeta^{-1}\mathcal{Z}_h(x))$.

This Lagrangian has the classical ground state at $\vec{x} = \vec{x}_*$ where $|\mathcal{Z}_h(x_*)| = \text{Re}[\zeta^{-1}\mathcal{Z}_h(x_*)]$, with the ground state energy $\text{Re}[\zeta^{-1}\mathcal{Z}_h(\infty)]$. We wish to elevate this, later, to $\mathcal{N} = 4$ quantum mechanics, so it is more convenient to separate out the ground state energy. Thus, our starting point is the bosonic Lagrangian,

$$\mathcal{L}_{\text{moduli}}^{\text{bosonic}} = \mathcal{L}^{\text{bosonic}} + \text{Re}[\zeta^{-1}\mathcal{Z}_h(\infty)] , \quad (2.21)$$

so that supersymmetric bound states would have zero energy. This also reproduces an analog of Denef's formula [15] for the probe dyons since,

$$\sum_A \frac{\langle \gamma_{c,A}, \gamma_h \rangle}{|\vec{x}_A - \vec{x}_*|} = \text{Im}[\zeta^{-1}\mathcal{Z}_h(\infty)] . \quad (2.22)$$

from Eq. (2.11) and $\text{Im}[\zeta^{-1}\mathcal{Z}_h(x_*)] = 0$. This is the same as (1.5) once we realize that total central charge $Z_T = Z_c + Z_h$ is dominated by Z_c since Z_h/Z_c is very small; ζ_T is approximately equal to ζ .

2.4 Fermionic Partners

We have derived a classical (thus purely bosonic) Lagrangian that describe the dynamics of a probe dyon in the background of the core state, with 3 bosonic collective coordinates per each probe dyon. Without much effort, we can further deduce

that each probe dyon will also come with 4 fermionic degrees of freedom, giving $4n$ fermionic variables as opposed to $3n$ bosonic variables.

The simplest way to see those four fermionic variables is to recall that a BPS particle, of a given charge, in $N = 2$ theory are at least in the half-hypermultiplet, with spin content

$$[1/2] \oplus 2[0] . \quad (2.23)$$

This spin content can be generated only if the dyon comes with a pair of complex fermionic degrees of freedom in a spin $1/2$ multiplet, which translates to four real fermionic coordinates. They are, when we consider the dyon in isolation, also Goldstino modes coming from the four supercharges broken by the BPS state. More generally, the probe dyon could be in a BPS multiplet of type,

$$[s] \otimes ([1/2] \oplus 2[0]) , \quad (2.24)$$

with an angular momentum multiplet $[s]$ of spin s , in which case $[s]$ typically arises because the probe dyon is itself a composite or has, otherwise, some internal light degrees of freedom. What matters for us is that we still have the same four fermionic collective coordinates whose coupling to the bosonic ones are tightly constrained by the $\mathcal{N} = 4$ supersymmetries.

When we consider the special limit of solitonic dyons in weakly coupled theories, this mismatch between the bosonic and the fermionic degrees of freedom can be understood easily [29, 30, 37]. Solitonic dyons arise there from excitation of a monopole soliton with particular $U(1)$ momenta turned on [31]. While the initial monopole soliton comes with four bosonic and four fermionic collective coordinates, one angular bosonic coordinate is traded away in favor of its conjugate momentum (which is physically the electric charge). This procedure, however, leaves the four fermionic coordinates intact. It has to be so, since the dyon is still BPS and the necessary half-hypermultiplet structure would be generated using all four of these fermionic degrees of freedom. Nor does this reduce the $\mathcal{N} = 4$ supersymmetry of the remaining dynamics, although their embedding into the underlying field theory is rotated in response to the new electromagnetic charges.

3 Massive $\mathcal{N} = 4$ Mechanics onto Moduli Space

An odd fact, when we consider a supersymmetric lift of the above Lagrangian for probe dyons, is that the low energy dynamics involves 3 bosonic collective coordinates for each probe dyon, yet, there should be 4 fermionic counterparts. Supersymmetry with mismatching bosonic and fermionic degrees of freedom is in principle possible for quantum mechanics because there is no notion of spin, but still construction of

such theories, especially with extended supersymmetry, was not widely studied. The only known example is certain (massless) class of supersymmetric nonlinear sigma models by Coles et. al. [21], which were later specialized in the context of extremely charged black holes of the same charges [22]. Neither of these studies considered massive versions, as needed here, however.

Similar situation existed a dozen years ago when low energy dynamics of solitonic monopoles were studied for weakly coupled $N = 2, 4$ Yang-Mills theories. The conventional massless moduli dynamics [32, 33, 34] with $4n$ -dimensional target manifolds without potential were found to be inadequate for dyons in generic Coulombic vacuum when the rank of the gauge group is two or larger [8]. The problem was the lack of potential terms in this older formulation. The low energy dynamics of monopoles had to be reformulated so that both the potentials and $\mathcal{N} = 4$ supersymmetry are manifest. Later, such massive $\mathcal{N} = 4$ quantum mechanics were found, simply by twisting supercharged by triholomorphic Killing vector fields on the moduli space [9, 11, 10, 12].^{#5} This lead to a whole machinery whereby dyon spectra in the weakly coupled limit of $N = 2, 4$ Yang-Mills theories were constructed explicitly [14]. See Ref. [37] for a broad overview of this development.

In this section, we wish to investigate how the new kind of classical low energy dynamics of section 2 can be also elevated to one with $\mathcal{N} = 4$ supersymmetry. We will find that massive $\mathcal{N} = 4$ supersymmetric mechanics with mismatching bosonic and fermionic degrees of freedom is possible and will, specifically, build a massive (i.e. with potential) supersymmetric Lagrangian with 3 bosonic coordinates and 4 fermionic coordinates. This restriction to the lowest possible target dimension simplifies the construction greatly, in part because the target manifold turned out to be conformally flat R^3 , and yet still good enough for deriving semi-primitive wall-crossing formula.^{#6}

3.1 Toy Model: Flat R^3 Target

As a toy model, let us pretend that the bosonic moduli space is flat R^3 and see how scalar and vector potentials on R^3 can be incorporated into the quantum mechanics in a manner consistent with four supercharges.

$\mathcal{N} = 1$ supersymmetry is easy to incorporate. We start with the usual transfor-

^{#5}Some related mathematical structures were first studied in Refs. [35] while its potential connection to dyons was previously hinted by Ref. [36].

^{#6}For generalization that can address many probe dyons with non-negligible mutual interactions, we need to consider higher dimensional target manifolds, which is left for a future work.

mation rule

$$\delta x^a = -i\epsilon\psi^a, \quad \delta\psi^a = \epsilon\dot{x}^a, \quad (3.1)$$

under which the following free Lagrangian that is invariant

$$\mathcal{L}^{(0)} = \frac{1}{2}\dot{x}^a\dot{x}^a + \frac{i}{2}\psi^a\dot{\psi}^a. \quad (3.2)$$

Since we are dealing with quantum mechanics, rather than a field theory, we can add any number of fermions, as long as we let them be invariant under the above supersymmetry transformation. As we will see shortly, however, extended supersymmetry would not leave this extra fermion intact.

For our purpose, one extra fermion λ is needed for each triplet of (x^a, ψ^a) , so we may start with

$$\mathcal{L}^{(0)} = \frac{1}{2}\dot{x}^a\dot{x}^a + \frac{i}{2}\psi^a\dot{\psi}^a + \frac{i}{2}\lambda\dot{\lambda}, \quad (3.3)$$

where

$$\delta\lambda = 0. \quad (3.4)$$

Incorporation of an external gauge field w on R^3 is equally easy. Adding a minimal coupling $-\dot{x}^a w_a$ to the Lagrangian and noting the supersymmetry transformation property,

$$\begin{aligned} \delta(-w_a\dot{x}^a) &= +i\epsilon\psi^a\dot{x}^b(\partial_a w_b - \partial_b w_a) + \text{total derivative} \\ &= +i\partial_a w_b(\epsilon\psi^a\dot{x}^b - \epsilon\psi^b\dot{x}^a), \end{aligned} \quad (3.5)$$

one finds a canceling term of type

$$\delta(+i\partial_a w_b\psi^a\dot{\psi}^b) = +i\partial_a w_b(\dot{x}^a\epsilon\psi^b - \dot{x}^b\epsilon\psi^a). \quad (3.6)$$

In summary, the following Lagrangian has $\mathcal{N} = 1$ supersymmetry

$$\mathcal{L} = \frac{1}{2}\dot{x}^a\dot{x}^a + \frac{i}{2}\psi^a\dot{\psi}^a + \frac{i}{2}\lambda\dot{\lambda} - w_a\dot{x}^a + i\partial_a w_b\psi^a\dot{\psi}^b. \quad (3.7)$$

In order to introduce the bosonic potential to the above model, we modify the transformation rule of the auxiliary fermion λ as

$$\delta x^a = -i\epsilon\psi^a, \quad \delta\psi^a = \epsilon\dot{x}^a, \quad \delta\lambda = \epsilon K, \quad (3.8)$$

upon which we find

$$\delta \left(\frac{1}{2} \dot{x}^a \dot{x}^a + \frac{i}{2} \psi^a \dot{\psi}^a + \frac{i}{2} \lambda \dot{\lambda} - w_a \dot{x}^a + i \partial_a w_b \psi^a \psi^b \right) = -i \epsilon \lambda \dot{x}^a \partial_a K . \quad (3.9)$$

The canceling term for this is

$$\begin{aligned} \delta \left(+ i \partial_a K \psi^a \lambda \right) &= i \epsilon \lambda \dot{x}^a \partial_a K - i \epsilon \psi^a K \partial_a K \\ &= i \epsilon \lambda \dot{x}^a \partial_a K + \delta x^a K \partial_a K , \end{aligned} \quad (3.10)$$

while one must add one more to close the transformation algebra,

$$\delta \left(- \frac{1}{2} K^2 \right) = - \delta x^a K \partial_a K . \quad (3.11)$$

In summary, the Lagrangian

$$\mathcal{L} = \frac{1}{2} \dot{x}^a \dot{x}^a + \frac{i}{2} \psi^a \dot{\psi}^a + \frac{i}{2} \lambda \dot{\lambda} - w_a \dot{x}^a - \frac{1}{2} K^2 + i \partial_a w_b \psi^a \psi^b + i \partial_a K \psi^a \lambda \quad (3.12)$$

has $\mathcal{N} = 1$ supersymmetry, for any K and w .

We eventually wish to formulate dyon dynamics with $\mathcal{N} = 4$ supersymmetries. For conventional supersymmetric quantum mechanics, this requires the target manifold to be $4n$ dimensional and hyperKähler, which is clearly inappropriate for our $3n$ dimensional target. Nevertheless, the BPS nature of the dyons and existence of BPS bound states implies that there should exist such an $\mathcal{N} = 4$ lift.

To find the relevant supersymmetries and the subsequent restrictions on the potentials, note that, since the number of bosons and the number of fermions mismatch by 3 to 4, we can organize the degrees of freedom using $SO(4) = SU(2)_L \times SU(2)_R$ algebra. Let the bosons, $x^a, a = 1, 2, 3$, transform as $(\mathbf{3}, \mathbf{1})$ representation while the fermions, (ψ^a, λ) , are naturally in $(\mathbf{2}, \mathbf{2})$ and better denoted as $\psi^m, m = 1, 2, 3, 4$ with $\psi^4 = \lambda$. Thus, a, b, \dots are the vector indices of $SU(2)_L$ while m, n, \dots are vector indices of $SO(4)$. The $\mathcal{N} = 4$ supersymmetries are then naturally in $(\mathbf{2}, \mathbf{2})$ under this $SO(4)$, since it should relate bosons to fermions. Thus the four supersymmetry transformation parameters will be denoted by ϵ_m .

A useful method of relating $SU(2)_L$ objects to $SO(4)$ object is to employ 't Hooft's self-dual symbol η_{mn}^a . Based on previous experience of embedding $SO(3) \simeq SU(2)_L$ into $SO(4)$, such as in Yang-Mills instanton construction, one can guess the following $\mathcal{N} = 4$ SUSY transformation rules

$$\delta x^a = i \eta_{mn}^a \epsilon^m \psi^n , \quad \delta \psi_m = \eta_{mn}^a \epsilon^n \dot{x}^a + \epsilon_m K , \quad (3.13)$$

with the 't Hooft self-dual symbol η defined as [38]

$$\eta_{bc}^a = \epsilon_{abc}, \quad \eta_{b4}^a = \delta_b^a = -\eta_{4b}^a . \quad (3.14)$$

which, for $\epsilon^4 \equiv \epsilon$, matches (3.8).

This suggests that the Lagrangian (3.12) can be extended to admit $\mathcal{N} = 4$ supersymmetries, if we can organize the fermion bilinears in terms of η symbol as

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^3 \dot{x}^a \dot{x}^a + \frac{i}{2} \sum_{m=1}^4 \psi^m \dot{\psi}^m + \frac{i}{2} \eta_{mn}^a \partial_a K \psi^m \psi^n - w_a \dot{x}^a - \frac{1}{2} K^2 , \quad (3.15)$$

which matches (3.12) if we impose

$$\epsilon^{abc} \partial_a K = \partial_b w_c - \partial_c w_b . \quad (3.16)$$

One can indeed show that the above Lagrangian is invariant under the $\mathcal{N} = 4$ SUSY transformation rules (3.13). This Lagrangian is manifestly invariant under $SU(2)_R$. The $SU(2)_L$ invariance is broken only to the extent that K breaks the rotational invariance. If K is spherically symmetric, for instance, the full $SO(4)$ symmetry would be restored.

Let us discuss the closure of the $\mathcal{N} = 4$ algebra. For bosonic variables, one can show

$$\delta_\zeta \delta_\epsilon x^a = -i \eta_{mn}^a \eta_{pn}^b \epsilon^m \zeta^p \dot{x}^b + i \eta_{mn}^a \epsilon^m \zeta^n K , \quad (3.17)$$

which implies

$$(\delta_\zeta \delta_\epsilon - \delta_\epsilon \delta_\zeta) x^a = -2i \epsilon^m \zeta^m \dot{x}^a . \quad (3.18)$$

Here we used the following identity $\eta_{mn}^a \eta_{pn}^b = \delta^{ab} \delta_{mp} + \epsilon^{abc} \eta_{mp}^c$. Let us now in turn consider the case of fermionic variables.

$$\begin{aligned} (\delta_\zeta \delta_\epsilon - \delta_\epsilon \delta_\zeta) \psi_m &= -2i \epsilon^n \zeta^n \dot{\psi}_m + i (\epsilon^n \zeta^m + \epsilon^m \zeta^n) \dot{\psi}^n + i \eta_{pq}^a \partial_a K (\epsilon^p \zeta^m + \epsilon^m \zeta^p) \psi^q , \\ &= -2i \epsilon^n \zeta^n \dot{\psi}_m , \end{aligned} \quad (3.19)$$

where for the last equality we used the equation of motion of ψ^m

$$\dot{\psi}^q + \eta_{qn}^a \partial_a K \psi^n = 0 . \quad (3.20)$$

We therefore conclude that the $\mathcal{N} = 4$ SUSY algebra is given by

$$\{Q_m, Q_n\} = 2\delta_{mn} H , \quad (3.21)$$

with the Hamiltonian H .

3.2 Toy Model with $\mathcal{N} = 1$ Superfields

We can write the above Lagrangian by introducing $\mathcal{N} = 1$ superspace with an anti-commuting coordinate θ . Following the notation in Ref. [22], we define the bosonic and the fermionic superfields as

$$\Phi^a = x^a - i\theta\psi^a, \quad \Lambda = i\lambda + i\theta b. \quad (3.22)$$

The supersymmetry generator and the supercovariant derivatives are then,

$$Q = \partial_\theta + i\theta\partial_t, \quad D = \partial_\theta - i\theta\partial_t. \quad (3.23)$$

Our toy model based on flat R^3 , with scalar and vector potentials, can be written in a superspace form as

$$\mathcal{L} = \int d\theta \left(\frac{i}{2} D\Phi^a \partial_t \Phi^a - \frac{1}{2} \Lambda D\Lambda + iK(\Phi)\Lambda - iw(\Phi)_a D\Phi^a \right). \quad (3.24)$$

Although only $\mathcal{N} = 1$ supersymmetry is manifest, we saw that $\mathcal{N} = 4$ supersymmetry will emerge if the condition $*dK = dw$ is imposed. This form of the Lagrangian is useful because it could be generalized to the curved moduli space almost immediately.

3.3 Massive $\mathcal{N} = 4$ Theory onto Conformally Flat R^3

Recall that, for a single probe dyon, there are three quantities that appears in the bosonic moduli dynamics. The scalar and the vector potentials, as we already incorporated into $\mathcal{N} = 4$ toy model above, and most crucially, the position-dependent mass term $|\mathcal{Z}_h|$ for the coordinates x^a . Thus, in addition to the above interaction terms, we wish to replace R^3 by a conformal flat R^3 whose metric is

$$g_{ab} = f\delta_{ab}, \quad (3.25)$$

with f later to be identified with $|\mathcal{Z}_h|$. In fact, as can be inferred from the massless version in Refs. [21, 22], $\mathcal{N} = 4$ supersymmetry restricts the three-dimensional metric to be conformally flat.

We defer detailed construction to appendix B, and simply state here that the desired Lagrangian, now with potentials, has the superspace form

$$\begin{aligned} \mathcal{L} = \int d\theta & \left(\frac{i}{2} f(\Phi) D\Phi^a \partial_t \Phi^a - \frac{1}{2} f(\Phi) \Lambda D\Lambda \right. \\ & \left. + \frac{1}{4} \epsilon_{abc} \partial_a f(\Phi) D\Phi^b D\Phi^c \Lambda + i\mathcal{K}(\Phi)\Lambda - i\mathcal{W}(\Phi)_a D\Phi^a \right), \end{aligned} \quad (3.26)$$

with the condition

$$\partial_a \mathcal{K} = \epsilon_{abc} \partial_b \mathcal{W}_c \quad (3.27)$$

imposed. In terms of component fields, this equals

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} f (\dot{x}^a \dot{x}^a + i \psi^m \nabla_t \psi^m) \\ & - \frac{1}{4 \cdot 4!} (2\partial^2 f - f^{-1}(\partial f)^2) \epsilon_{mnpq} \psi^m \psi^n \psi^p \psi^q \\ & - \frac{1}{2f} \mathcal{K}^2 - \mathcal{W}_a \dot{x}^a + \frac{i}{2} f^{1/2} \partial_a (f^{-1/2} \mathcal{K}) \eta_{mn}^a \psi^m \psi^n, \end{aligned} \quad (3.28)$$

where the covariant derivative for fermions is given by

$$\nabla_t \psi^m = \dot{\psi}^m + \frac{1}{2} \epsilon_{abc} \dot{x}^a f^{-1} \partial_b f \eta_{mn}^c \psi^n. \quad (3.29)$$

As in the flat case, the degrees of freedom and the supercharges are cataloged by $SO(4) = SU(2)_L \times SU(2)_R$ algebra, and the Lagrangian is manifestly invariant under $SU(2)_R$. The $SU(2)_L$ keeps track of how f and \mathcal{K} (and thus \mathcal{W} also) transform under spatial rotations, and become a symmetry whenever these quantities are rotationally invariant.

This $SO(4)$ structure and $SU(2)_R$ symmetry tells us an extended $\mathcal{N} = 4$ supersymmetry exists, as in the flat R^3 example. It is not difficult to see that

$$\begin{aligned} \delta_\epsilon x^a &= i \eta_{mn}^a \epsilon^m \psi^n, \\ \delta_\epsilon \psi_m &= \eta_{mn}^a \epsilon^n \dot{x}^a + \epsilon_m b, \end{aligned} \quad (3.30)$$

with four Grassman parameters ϵ^m leaves the Lagrangian invariant. The only difference from the flat case, (3.17), is that K is replaced by its generalized form, namely on-shell value of the $\mathcal{N} = 1$ auxiliary field b ,

$$b = \frac{1}{f} \left(\mathcal{K} + \frac{i}{4} \eta_{pq}^a \partial_a f \psi^p \psi^q \right). \quad (3.31)$$

The superalgebra remains the same as the flat case,

$$\{Q_m, Q_n\} = 2\delta_{mn} H, \quad (3.32)$$

where we denoted the four supercharges by Q_m as before and the Hamiltonian by H . For completeness, we also record the classical form of the Hamiltonian,

$$\begin{aligned} H_{classical} = & \frac{1}{2f} \pi^a \pi^a + \frac{1}{4 \cdot 4!} (2\partial^2 f - f^{-1}(\partial f)^2) \epsilon_{mnpq} \psi^m \psi^n \psi^p \psi^q \\ & + \frac{1}{2f} \mathcal{K}^2 - \frac{i}{2} f^{1/2} \partial_a (f^{-1/2} \mathcal{K}) \eta_{mn}^a \psi^m \psi^n, \end{aligned} \quad (3.33)$$

with the covariantized momenta

$$\pi^a = p_a + \mathcal{W}_a - \frac{i}{4} \epsilon_{abc} \partial_b f \eta_{mn}^c \psi^m \psi^n . \quad (3.34)$$

The quantum Hamiltonian differs from this by normal ordering issue, and can also be found in appendix B.

4 Quantum BPS States near WMS

4.1 $\mathcal{N} = 4$ Low Energy Dynamics of Dyons near MSW

Let us stop here and ask under what circumstances we actually expect to see a sensible low energy dynamics of dyons to appear. The old setting based on dyons as quantum bound states of excited magnetic solitons was possible by resorting to weakly coupled regime. There, the basic requirements was that the energy due to electric charges and also due to motion of the solitons are of higher order. Thus, the reduction to quantum mechanics is controlled two small parameters; typical speed of the magnetic soliton and the electric coupling constant [32].

Here, however, we are here dealing with dyons of generic charges at generic coupling, and must find different criteria to justify reduction to low energy quantum mechanics. Note that the weak coupling requirement and the small speed requirement of old moduli dynamics is in fact interrelated. That happens was that the moduli dynamics of $N = 2$ and $N = 4$ monopoles usually acquire a bosonic potential of order e^2 , so for typical states the small electric coupling is necessary to ensure small velocities.

In the present low energy dynamics of probe dyons around a core state, the size of the potential is instead controlled by how far are the phases of central charges of core and halo particles are aligned. Thus, by staying very near MSW, we have a good control over the potentials. Furthermore, the massgap between this sector and the rest is also substantial, and controls possible interference from other charged particles.^{#7} So it is the proximity to the MSW and also the mass ratio of the two parts that now control the reduction to the low energy quantum mechanics.

With this mind, we compare (2.21) against the supersymmetric Lagrangian (3.28).

^{#7}The latter is easiest to see when the small mass ratio is achieved by being near a singular point of the vacuum moduli space. The relevant coupling that governs the interaction of the field theory would be a dualized coupling which becomes small as the singular point is approached.

One can see the supersymmetric uplift may work only if

$$f = |\mathcal{Z}_h|, \quad \frac{1}{2f}\mathcal{K}^2 = |\mathcal{Z}_h| - \text{Re}[\zeta^{-1}\mathcal{Z}_h], \quad \vec{\nabla} \times \vec{\mathcal{W}} = \vec{\nabla} (\text{Im}[\zeta^{-1}\mathcal{Z}_h]) . \quad (4.1)$$

but the requisite $\mathcal{N} = 4$ relationship between \mathcal{K} and \mathcal{W} , $*d\mathcal{K} = d\mathcal{W}$, is not yet apparent. Thankfully, this condition is satisfied precisely when the criteria for the low energy approximation are met, as we described above.

To see the latter, write $\zeta^{-1}\mathcal{Z}_h = |\mathcal{Z}_h|e^{i\beta}$. Near the wall of marginal stability, the angle β at spatial infinity is very small whereas its value at classical vacuum is 0. Recall that the bound states we wish to find and count are all peaked at the classical vacuum manifold. This allows us to expand relevant quantities in small β . As we move closer to charge centers, \vec{x}_A 's, β can grow again but the precisely form of the background at such charge centers are not to be trusted and also happily immaterial for our purpose of finding BPS bound states. Therefore, we take the value of β to be small everywhere and find

$$\mathcal{K}^2 = 2|\mathcal{Z}_h|^2(1 - \cos \beta) \simeq |\mathcal{Z}_h|^2\beta^2 \simeq |\mathcal{Z}_h|^2(\sin \beta)^2 = (\text{Im}[\zeta^{-1}\mathcal{Z}_h])^2 . \quad (4.2)$$

Thus, for all practical purpose, we may identify $\mathcal{K} = \text{Im}[\zeta^{-1}\mathcal{Z}_h]$ and the $\mathcal{N} = 4$ requirement (3.27) is obeyed automatically. This completes the derivation of $\mathcal{N} = 4$ moduli dynamics in (3.28) of a probe dyon in a given core state background.

The function \mathcal{K} can be generally written, from (2.11), as

$$\mathcal{K} = \mathcal{K}_0 - \sum_A \frac{\langle \gamma_{c,A}, \gamma_h \rangle}{|\vec{x} - \vec{x}_A|} , \quad (4.3)$$

with $\gamma_{c,A}$ centers of the core states at \vec{x}_A and also $\mathcal{K}_0 \equiv \text{Im}[\zeta^{-1}\mathcal{Z}_h(\infty)]$. Details of $f = |\mathcal{Z}_h|$ won't matter much for the purpose of constructing bound states, it turns out, as long as we keep track of its singular behaviors at charge centers.

Before we start the detailed analysis, let us note again that the semiclassical core state here is not really a good representation very near its charge center(s), where the non-Abelian nature of the states becomes relevant. Naturally, the low energy dynamics of probe dyons is plagued by the same issue. However, this hardly matters near MSW because the bound state (if any) would be very large and determined entirely by Abelian part of the low energy field theory: Whatever singularity at Coulombic centers cannot alter such wavefunction significantly, as long as we impose the boundary condition at centers intelligently enough. This should become more obvious when we discuss actual bound state wavefunctions in section 4.3. For this reason, we may as well take the above form of \mathcal{K} , f , etc literally, and consider supersymmetric bound states thereof, with some care given to the boundary condition of the wavefunctions at centers \vec{x}_A .

4.2 Quantization and Supercharges

Let us start with the canonical commutators. The conjugate momenta of bosons are denoted as p_a ,

$$[p_a, x^b] = -i\delta_a^b, \quad (4.4)$$

whereas the normalized fermions, $\hat{\psi}^m \equiv f^{1/2}\psi^m$, are more convenient for writing out the remaining canonical commutators,

$$\{\hat{\psi}^m, \hat{\psi}^n\} = \delta^{mn}, \quad [p_a, \hat{\psi}^m] = 0 = [x^a, \hat{\psi}^n]. \quad (4.5)$$

With this we can now write the four supercharges as

$$Q_m = -\eta_{mn}^a \psi^n (p_a + \mathcal{W}_a) + \frac{i}{4} \eta_{mn}^a f^{-1} \partial_a f \psi^n + \frac{i}{4} \partial_a f \eta_{pq}^a \psi^{[p} \psi^q \psi^m] + \mathcal{K} \psi^m. \quad (4.6)$$

For the proof that these are right supercharges, see appendix B. In particular, the supercharge associated with ϵ^4 is

$$Q = Q_4 = \psi^a (p_a + \mathcal{W}_a) - \frac{i}{4} f^{-1} \partial_a f \psi^a + \frac{i}{4} \partial_a f \epsilon_{abc} \psi^b \psi^c \lambda + \lambda \mathcal{K}. \quad (4.7)$$

Since the superalgebra implies $\{Q_m, Q_n\} = 2\delta_{mn}H$, the ground state of the system can be found by demanding that it be annihilated by Q_4 .

4.3 BPS Bound States and Marginal Stability

The canonical commutator of the fermions

$$\{\hat{\psi}^m, \hat{\psi}^n\} = \delta^{mn} \quad (4.8)$$

is a Clifford algebra which can be represented by Dirac matrices,

$$\sqrt{2} \hat{\psi}^a = \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix}, \quad \sqrt{2} \hat{\psi}^4 = \gamma^4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (4.9)$$

and wavefunctions can be regarded as 4-component spinors on R^3 . Also useful is the chirality operator

$$\Gamma \equiv \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.10)$$

Under the above representation, one of supercharge Q_4 now has a simple form,

$$\sqrt{2f} Q_4 = \gamma^a (p_a + \mathcal{W}_a) - \frac{i}{2} (\partial_a \log f) \gamma^a \frac{1 - \Gamma}{2} + \mathcal{K} \gamma^4, \quad (4.11)$$

or more explicitly,

$$\sqrt{2f} Q_4 = \begin{pmatrix} 0 & \sigma \cdot (p + \mathcal{W}) + i\mathcal{K} \\ \sigma \cdot (p + \mathcal{W}) - i\sigma \cdot \partial(\log f^{1/2}) - i\mathcal{K} & 0 \end{pmatrix} . \quad (4.12)$$

We wish to find supersymmetric ground states, $Q_4\Psi = 0$. Since $\mathcal{H}\Psi = 0$ then, such states would actually preserve all four supercharges. Such states are then automatically BPS with respect to the $N = 2$ field theory with the energy $\text{Re}[\zeta^{-1}Z_h(\infty)]$, as can be seen from (2.21), not counting the core state energy.

Write the four-component wavefunction Ψ as

$$\Psi = \begin{pmatrix} f^{-1/2}\mathcal{U} \\ \mathcal{V} \end{pmatrix} , \quad (4.13)$$

upon which two component wavefunctions \mathcal{U} and \mathcal{V} obey

$$\begin{aligned} (\sigma \cdot (p + \mathcal{W}) - i\mathcal{K})\mathcal{U} &= 0, \\ (\sigma \cdot (p + \mathcal{W}) + i\mathcal{K})\mathcal{V} &= 0, \end{aligned} \quad (4.14)$$

With the supersymmetry condition $d\mathcal{K} = *d\mathcal{W}$, it is easy to see that the first equation cannot have a normalizable solution while the second may. Denoting the respective operators as \mathcal{D}_\pm ,

$$\mathcal{D}_\mp \mathcal{D}_\pm = (p + \mathcal{W})^2 + \mathcal{K}^2 + \sigma^a (\partial_a \mathcal{K} \pm \partial_a \mathcal{K}) , \quad (4.15)$$

which shows that $\mathcal{D}_+ \mathcal{D}_-$ is a positive definite operator while $\mathcal{D}_- \mathcal{D}_+$ is not. Only the latter can have zero modes. Thus, we arrived at the conclusion that the counting of BPS bound states between the core dyon and the probe dyon becomes that of counting normalizable two-component zero modes \mathcal{V} of the operator \mathcal{D}_+ , with the final form of the BPS bound state

$$\Psi = \begin{pmatrix} 0 \\ \mathcal{V} \end{pmatrix} , \quad (4.16)$$

with $\mathcal{D}_+ \mathcal{V} = 0$.

It is illuminating to solve this equation for the particular case of spherically symmetry core state. The vector potential would be that of a Dirac monopole, so we denote

$$\mathcal{W} = -gA_{\text{Dirac}}, \quad g = -\langle \gamma_c, \gamma_h \rangle, \quad A_{\text{Dirac}} = -\cos\theta d\phi , \quad (4.17)$$

from which follows the scalar potential

$$\mathcal{K} = \mathcal{K}_0 + \frac{g}{r} . \quad (4.18)$$

In this case $SU(2)_L$ also becomes a symmetry, allowing an explicit solution to the bound state problem. The number g is half-integer quantized, as dictated by the Dirac quantization of this quantum mechanics, and also from its field theory origin as the Schwinger product of the two quantized charge vectors. The angular and spin part of the wavefunction is classified by spinorial monopole spherical harmonic tensor. The lowest possible angular momentum would be than $j = |g| - 1/2$, since the charge interacting with such a Dirac monopole, \mathcal{W} , is endowed with a well-known angular momentum $-g\hat{r}$. Tensoring with the intrinsic spin $1/2$, the minimum possible value $j = |g| - 1/2$ follows.

Denoting the corresponding the lowest-lying two-component angular momentum eigen-states $\eta_{j=|g|-1/2,m}$ of $SO(3) \simeq SU(2)_L$ we rely on Kazama et.al. [39] for reduction of the above to the radial equation,

$$\mathcal{V} = h(r)\eta_{j=|g|-1/2,m}, \quad \left(-i\frac{g}{|g|} \times \left[\frac{d}{dr} + \frac{1}{r} \right] + i\mathcal{K}(r) \right) h(r) = 0. \quad (4.19)$$

Integrating the latter equation, we find

$$h(r) = \frac{1}{r} \exp \left(\frac{g}{|g|} \int^r \mathcal{K}(r) \right) = C r^{|\langle \gamma_c, \gamma_h \rangle|^{-1}} \exp \left(-[\text{sgn}(\langle \gamma_c, \gamma_h \rangle) \cdot \mathcal{K}_0] \cdot r \right), \quad (4.20)$$

with the normalization constant C . Note that this gives a normalizable ground state if and only if the half-integer-quantized $\langle \gamma_c, \gamma_h \rangle$ is not zero and has the same sign as $\mathcal{K}_0 = \text{Im}[\zeta^{-1}\mathcal{Z}_h(\infty)]$.^{#8} The latter condition is also reflected on the fact that the probability density of this wavefunction is peaked at radial size

$$\frac{\langle \gamma_c, \gamma_h \rangle}{\text{Im}[\zeta^{-1}\mathcal{Z}_h(\infty)]}, \quad (4.21)$$

which, for a single-center core state, exactly mirrors the classical orbit radius in (2.22).

The sign of $\langle \gamma_c, \gamma_h \rangle$ is determined by the charges of the core state and the probe state, and does not change as we move along the vacuum moduli space. However, the $\mathcal{K}_0 = \text{Im}[\zeta^{-1}\mathcal{Z}_h(\infty)]$ does change its sign across the marginal stability wall between the core state and the probe state. Classically, this happens because $g\mathcal{K}_0 < 0$ would make the potential repulsive. The upshot is that the BPS bound states of one side where $\langle \gamma_c, \gamma_h \rangle / \text{Im}[\zeta^{-1}\mathcal{Z}_h(\infty)] > 0$ disappear as we move to the other side where $\langle \gamma_c, \gamma_h \rangle / \text{Im}[\zeta^{-1}\mathcal{Z}_h(\infty)] < 0$, as was originally found in the supergravity setting.

With this exercise, we learned a few things:

^{#8} L^2 normalizability requirement from $r = 0$ region is satisfied as long as $|\langle g_c, \gamma_h \rangle|$ is not zero, so does not impose additional restriction.

- Normalizable bound state between the core state and the probe state is realized only when the Schwinger product of the two charge is nonzero.
- Normalizable bound state between the core state and the probe state is realized only when the Schwinger product of the two charge is of the same sign relative to the value of $\text{Im}[\zeta^{-1}\mathcal{Z}_h]$ at spatial infinity.
- When such normalizable states exist, the degeneracy is $2j + 1 = 2|\langle\gamma_c, \gamma_h\rangle|$.

Much of the above statements are properties of a Dirac operator with \mathcal{D}_\pm as the chiral and the anti-chiral parts; there must be an index theorem associated with them.

In fact, the structure of the operators are essentially that of an electrically charged fermionic field around the magnetic monopole, except that we do not see the non-Abelian structure that regulate the short-distance behavior of the core state. Similar issues in the context of quantization in the backgrounds of non-Abelian monopoles vs. Dirac monopoles (or more precisely Wu-Yang monopoles [40]) have been studied in depth decades ago, where it was found that with proper boundary condition at origins of the latter, behaviors of the two are essentially the same [41]. The boundary condition is constrained by the requirement that the Dirac operator constructed out of \mathcal{D}_\pm should be Hermitian, which is known in the literature as the self-adjoint extension.

This is related to the fact that, even though the two potentials of the quantum mechanics are singular at origin, the wavefunctions found are regular everywhere and in particular suppressed strongly at origin. If we attempted to solve for $\mathcal{D}_-\mathcal{U} = 0$, the radial eigen-function of \mathcal{U} would have the behavior $r^{-|\langle\gamma_c, \gamma_h\rangle|-1}$ at origin and is clearly unacceptable. This again shows that only \mathcal{D}_+ can have a solution. In particular, the supersymmetric bound state are trustworthy even though the quantum mechanics itself would be corrected, at small r , by non-Abelian nature of such objects.

Therefore, the index problem of the above operator is on par with that of zero mode problems around non-Abelian monopoles; the Callias index theorem [42, 29, 30] should apply. We thus anticipate that the number of zero energy bound states is additive; when the core state is composed of many centers of charges $\gamma_{c,A}$ with $\langle\gamma_{c,A}, \gamma_h\rangle\mathcal{K}_0 > 0$, the number of the bound state of the probe dyon is the naive one,

$$2|\langle\gamma_c, \gamma_h\rangle| = \left| \sum_A 2\langle\gamma_{c,A}, \gamma_h\rangle \right|, \quad (4.22)$$

since $\gamma_c = \sum_A \gamma_{c,A}$.

5 Wall-Crossing from Moduli Dynamics

5.1 Primitive Wall-Crossing: $\gamma_c + \gamma_h$

So far, we ignored the precise supermultiplet structures; Our approximation allowed us to treat the supermultiplet structure of the core state as a separate sector, while we extracted only partial sector of the probe dyons which would have been responsible for building a half-hypermultiplet. More generally, the probe dyon can come with higher spin states, such as $N = 2$ vector multiplet or higher, so we may decompose the Hilbert space of the combined core-probe system as

$$\mathcal{H}_{\text{core}} \otimes \mathcal{H}_{\text{probe}}^{\text{reduced}} \otimes \mathcal{H}_{\text{moduli dynamics}} . \quad (5.1)$$

The reduced Hilbert space denotes part of the free Hilbert space of a BPS particle that multiplies the half-hypermultiplet,

$$\mathcal{H} = \mathcal{H}^{\text{reduced}} \otimes ([1/2] \oplus 2[0]) . \quad (5.2)$$

When the probe dyon is in the half-hypermultiplet,^{#9} $\mathcal{H}_{\text{probe}}^{\text{reduced}}$ would have only one state, while in the vector multiplet, it would be the angular momentum 1/2 Hilbert space, etc.

The decomposition (5.1) can be understood easily. The core part of the Hilbert space is inert, so can be treated as non-dynamical. Of the probe, the half-hypermultiplet part are generated by the universal would-be Goldstino modes which become no longer free due to the presence of the core state. Instead they participate in the moduli dynamics we constructed and thus belong to $\mathcal{H}_{\text{moduli dynamics}}$. Note that these four modes would become free at $r = \infty$, regaining its nature as Goldstino. The remaining part $\mathcal{H}_{\text{probe}}^{\text{reduced}}$ accounts for extra degeneracies and spin content of the probe supermultiplet, which should represent additional structure on top of the low energy dynamics.

On the other hand, the second helicity trace (1.3), which is the relevant index for $N = 2$ theories, takes value

$$\Omega([j] \otimes ([1/2] \oplus 2[0])) = (-1)^{2j}(2j+1) \quad (5.3)$$

for the irreducible angular momentum multiplet $[j]$, and can also be expressed as

$$\Omega(\mathcal{H}) = \text{tr}_{\mathcal{H}^{\text{reduced}}}(-1)^{2j_3} . \quad (5.4)$$

^{#9}Recall that usual hypermultiplet forms when the CTP conjugate states are taken into account.

The degrees of freedom for the core state does not participate in the dynamics, so we have the decomposition

$$\begin{aligned}
& \Omega \left(\mathcal{H}_{\text{core}} \otimes \mathcal{H}_{\text{probe}}^{\text{reduced}} \otimes \mathcal{H}_{\text{moduli dynamics}} \right) \\
&= \Omega \left(\mathcal{H}_{\text{core}} \right) \times \text{tr}_{\mathcal{H}_{\text{probe}}^{\text{reduced}}} (-1)^{2j_3} \times \text{tr}_{\mathcal{H}_{\text{moduli dynamics}}} (-1)^{2J_3} \\
&= \Omega \left(\mathcal{H}_{\text{core}} \right) \times \Omega \left(\mathcal{H}_{\text{probe}} \right) \times \text{tr}_{\mathcal{H}_{\text{moduli dynamics}}} (-1)^{2j_3} .
\end{aligned} \tag{5.5}$$

Combining with the supersymmetric bound state we found above, this reproduces the primitive wall-crossing formula of Denef,

$$\Delta\Omega(\gamma_c + \gamma_h) = -(-1)^{2|\langle\gamma_c, \gamma_h\rangle|} 2|\langle\gamma_c, \gamma_h\rangle| \Omega(\gamma_c) \Omega(\gamma_h) . \tag{5.6}$$

5.2 Semi-Primitive Wall Crossing: $\gamma_c + n\gamma_h$

The semi-primitive wall-crossing formula of Denef and Moore conjectures how many BPS states of charge $\gamma_c + n\gamma_h$ appears across a MSW, for positive integer n . In order to compute the degeneracies of such states we must consider n number of γ_h charges in the core state background of γ_c . The Lagrangian would be

$$\mathcal{L} = \sum_{i=1}^n \mathcal{L}_{(i)} + \mathcal{L}_{hh} , \tag{5.7}$$

where $\mathcal{L}_{(i)}$ denotes the one-particle Lagrangian for i -th probe dyon, all of which are of the identical form. \mathcal{L}_{hh} captures the interaction among (identical) probe particles.

In our approximation, the latter can be ignored as long as the charges are such that

$$|\langle\gamma_c, \gamma_h\rangle| \gg |\langle\gamma_h, \gamma'_h\rangle| . \tag{5.8}$$

In particular this is the case if the probe charges are all mutually local, e.g., the same or proportional to each other. Then, the latter term \mathcal{L}_{hh} represent the second order correction to the former's first order form and can be safely ignored. The only nontrivial remnant is the matter of statistics, as in any quantum mechanics of many identical particles.

In addition, there is also a logical possibility that one-particle BPS states of non-primitive charge $k\gamma_h$ exist. In supergravity, such states are always there, since black holes can have any quantized charges. In field theory setting, the situation is a little unclear. In five dimensions, multi-instanton bound state do exist in the maximally supersymmetric Yang-Mills theory as quantum one-particle states. However, they

are tied to the UV completion of this theory which is the mysterious $(2, 0)$ theories. In the more familiar four-dimensional Yang-Mills setting, we are yet to see such an example. Nevertheless, we will include the possibility that the probe dyon of our moduli dynamics is non-primitive. Then, counting the degeneracy of the bound states $\gamma_c + n\gamma_h$ is basically identical to partition of $n\gamma_h$ into identical halo particles of $n\gamma_h = (\sum_i m_i k_i)\gamma_h$ with some cares on the statistics of each dyon of charge $k_i\gamma_h$. If it turns out that such non-primitive states do not exist,^{#10} we may simply set $\Omega(k\gamma_h) = 0$ for $k \geq 2$.

The question of statistics lead us to consider the intrinsic spin of the individual probe particle in the moduli dynamics. While the quantum mechanics by itself won't tell us about statistics of the particle, we can invoke the usual spin-statistics relation and instead ask about the spin. Recall that the canonical commutators,

$$\{\hat{\psi}^m, \hat{\psi}^n\} = \delta^{mn} , \quad (5.9)$$

implies that the spatial rotation generators of $SU(2)_L$ acting on the wavefunction are

$$-\frac{i}{4} [\hat{\psi}^a, \hat{\psi}^b] - \frac{i}{4} \epsilon_{abc} [\hat{\psi}^c, \hat{\psi}^4] = \frac{1}{2} \epsilon_{abc} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^a \end{pmatrix} . \quad (5.10)$$

This shows that the 4-component wavefunction, Ψ , consists of a single spin doublet \mathcal{V} in the lower half and a pair of spin singlet states combined into the upper half part, \mathcal{U} . Recall that the bound states can appear only in the \mathcal{V} sector; the supercharge Q_4 is effectively positively definite on \mathcal{U} as we saw in section 4.3. Therefore the BPS bound state of a (half-)hypermultiplet probe and the core always involve of a spin $1/2$ wavefunction.

More generally, the probe might be in a bigger multiplet, where $\mathcal{H}_{\text{probe}}^{\text{reduced}}$ is also part of the data that enters the probe dynamics although we simply factored it out. Taking into account the latter, we can see that the probe particle can be seen as a particle of spin content in the moduli quantum mechanics

$$\mathcal{H}_{\text{probe}}^{\text{reduced}} \otimes ([1/2] \oplus 2[0]) , \quad (5.11)$$

but the BPS bound state appears only in the sector $\mathcal{H}_{\text{probe}}^{\text{reduced}} \otimes [1/2]$. For example, if $\mathcal{H}_{\text{probe}}^{\text{reduced}} = [S]$, the total spin of the probe dyon that is involved in the bound state formation is $S \pm 1/2$. Therefore, *as far as supersymmetric bound state formation*

^{#10}The result of the previous section is suggestive in this regard. The bound states exist only if the Schwinger product of the two constituent charges are nonzero. Even if we take into account the finite core mass, we expect that a single-particle bound state of type $k\gamma + \gamma$ probably does not exist, which in induction suggests absence of the state of charge $k\gamma$ for $k \geq 2$ altogether. An interesting question is how this feature is modified in the realm of supergravity, where black holes of large non-primitive charges appears.

goes, that the probe dyon can be treated as if it is Boson or Fermion for $2S$ odd or even, respectively.

Such assignment of statistics is precisely what we expect on the field theory ground: Note that $S = 0$ correspond to the hypermultiplet while $S = 1/2$ to the vector multiplet. When one construct BPS dyons in the weakly coupled theory, the simplest method is to excite massive electrically charged and L^2 -normalizable modes around magnetic soliton [8]. When the charged field is in the hypermultiplet, the relevant excitations arise all from the Dirac field and the Fermi statistics rule when we try to construct the dyons. For a vector multiplet, additional modes arise both from the vector field, so the Bosonic statistics become dominant. This naive construction works verbatim for $N = 4$ Yang-Mills theories, while for $N = 2$ only slightly modified (i.e., degeneracy shift by unit) as seen from more rigorous index computation [14, 17]. When we phrase the $N = 2$ result in terms of vector multiplet contributions vs. hypermultiplet contributions, we see the above statistics assignment emerging.

Interestingly, this statistics is correlated with the sign of index Ω of the probe dyon since

$$\Omega\left[[S] \otimes ([1/2] \oplus 2[0])\right] = (-1)^{2S}(2S+1) . \quad (5.12)$$

Thus, in the context of our probe moduli dynamics, probe dyons with positive Ω should behave as Fermions, while probe dyons with negative Ω should behave as Bosons. More generally, $\mathcal{H}_{\text{probe}}^{\text{reduced}}$ can be a direct sum of more than one spin sectors. We write

$$\mathcal{H}_{\text{probe}}^{\text{reduced}} = \oplus_{\sigma}[S_{\sigma}] = \mathbf{R}_+ \oplus \mathbf{R}_- , \quad (5.13)$$

with \mathbf{R}_{\pm} denoting the decomposition according to the sign $(-1)^{2S_{\sigma}}$. Thus,

$$\Omega_{\text{probe}} = \dim \mathbf{R}_+ - \dim \mathbf{R}_- . \quad (5.14)$$

For the purpose of the moduli quantum mechanics here, then, we effectively have $\dim \mathbf{R}_+$ Fermions and $\dim \mathbf{R}_-$ Bosons of the same probe charge.

Once this statistics issue is cleared, one can construct the generating function for the index $\Omega(\gamma_c + n\gamma_h)$ as follows

$$\sum_{n=0}^{\infty} \Omega(\gamma_c + n\gamma_h) q^n = \Omega(\gamma_c) \cdot \text{Tr} \left[(-1)^{2J_3} q^N \right] . \quad (5.15)$$

We used here the notation Tr to emphasize that it is performed also over the dyons of various charges $k\gamma_h$ as well as over the individual Fock space with the number operator N that counts the multiple probe dyons of the same charge. Let us split the number operator $N = \sum_{k, j_{\text{ext}}, j_{\sigma}^3} k N_{k, j_{\text{ext}}, j_{\sigma}^3}^B + \sum_{k, j_{\text{ext}}, j_{\sigma}^3} k N_{k, j_{\text{ext}}, j_{\sigma}^3}^F$ with N^B for bosons

and N^F for fermions. Here $|j_{\text{ext}}^3| \leq |\langle \gamma_c, k\gamma_h \rangle| - \frac{1}{2}$ and $|j_\sigma^3| \leq S_\sigma$. The relevant trace then becomes

$$\begin{aligned} & \text{Tr} \left[(-1)^{2J_3} q^N \right] \\ &= \sum_{N_{k,j_{\text{ext}}^3,j_\sigma^3}^{B/F}} (-1)^{\sum_{k,j_{\text{ext}}^3,j_\sigma^3} (2j_{\text{ext}}^3 + 2j_\sigma^3) \left(N_{k,j_{\text{ext}}^3,j_\sigma^3}^B + N_{k,j_{\text{ext}}^3,j_\sigma^3}^F \right)} q^{\sum_{k,j_{\text{ext}}^3,j_\sigma^3} k \left(N_{k,j_{\text{ext}}^3,j_\sigma^3}^B + N_{k,j_{\text{ext}}^3,j_\sigma^3}^F \right)}, \end{aligned}$$

which can be summed explicitly as

$$\begin{aligned} & \prod_k \prod_{j_{\text{ext}}^3, j_\sigma^3} \left(\sum_{N^B=0}^{\infty} \left[(-1)^{2k|\langle \gamma_c, \gamma_h \rangle|} q^k \right]^{N^B} \right) \cdot \prod_k \prod_{j_{\text{ext}}^3, j_\sigma^3} \left(\sum_{N^F=0}^1 \left[-(-1)^{2k|\langle \gamma_c, \gamma_h \rangle|} q^k \right]^{N^F} \right) \\ &= \prod_k \left[1 - (-1)^{2k|\langle \gamma_c, \gamma_h \rangle|} q^k \right]^{\dim(j_{\text{ext}}) \cdot (\dim(\mathbf{R}_+) - \dim(\mathbf{R}_-))} \\ &= \prod_k \left[1 - (-1)^{2k\langle \gamma_c, \gamma_h \rangle} q^k \right]^{2|\langle \gamma_c, k\gamma_h \rangle| \Omega(k\gamma_h)}. \end{aligned} \tag{5.16}$$

It shows that the generating function is

$$\sum_{n=0} \Omega(\gamma_c + n\gamma_h) q^n = \Omega(\gamma_c) \prod_{k=1} \left[1 - (-1)^{2k\langle \gamma_h, \gamma_c \rangle} q^k \right]^{2k|\langle \gamma_h, \gamma_c \rangle| \Omega(k\gamma_h)}. \tag{5.17}$$

This is precisely the semi-crossing wall-crossing formula conjectured by Denef and Moore [18], provided that the one-particle states of charge $\gamma_c + n\gamma_h$ are absent on the other side of the wall. Note that the latter assumption is guaranteed by our moduli dynamics. Thus, by staying near the walls of marginal stability and adjusting the probe dyon to be much lighter than the core, we have derived the semi-primitive wall-crossing formulae from the first principle.

6 Conclusion and Discussion

We have derived a $\mathcal{N} = 4$ supersymmetry low energy dynamics that govern probe dyons interacting with relatively heavy core states, in the long distance approximation. The proximity of the Coulomb vacuum to the marginal stability wall acts as a crucial control parameter that allows this non-relativistic quantum mechanical description, and we were able to reproduce the conjectured primitive and semi-primitive wall-crossing formulae for Seiberg-Witten theory dyons.

An important technological step here was to incorporate the potential energy of the probe particles, due to the core state, into the supersymmetric quantum mechanics. Because the latter comes with different bosonic and fermionic degrees of freedom, a nonconventional form of the supersymmetric low energy theory emerged, but in a manner consistent with the BPS structure of the underlying $N = 2$ field theory in question.

As we mentioned early on, our approximation scheme was inspired by the notion of framed BPS state in presence of a line operator. See Appendix C for a short review on line operator in relation to the wall-crossing. In a sense the line operator provides a setting where our computation would become an exact description and can aid evaluation of the line operator expectation values. The vacuum expectation of line operator is in effect a $(-1)^F$ weighted trace over the Hilbert space with a particular charge object Γ inserted as an external object,

$$\langle L_\Gamma \rangle = \text{Tr}_{\mathcal{H}_\Gamma} \left[(-1)^F e^{-2\pi R \hat{H}} \right], \quad \hat{H} = \{ \mathcal{Q}_\zeta^\dagger, \mathcal{Q}_\zeta \}, \quad (6.1)$$

where \mathcal{Q}_ζ denote the supercharges preserved by the line operator. It was conjectured that this observable can be expanded into

$$\langle L_\Gamma \rangle_{\gamma_h} = \sum_{\gamma_h} \Omega(\Gamma + \gamma_h) \mathcal{X}_{\gamma_h}, \quad (6.2)$$

where \mathcal{X}_{γ_h} 's are the Darboux coordinates of [44]. The semi-classical analysis on the conjectured form of $\langle L_\Gamma \rangle$ would be interesting and illuminating as in Ref. [43]. As noted by Gaiotto et.al [44, 45], this asserts the much needed continuity property of \mathcal{X} 's over the vacuum moduli space that plays a central role justifying KS formalism in the context of $N = 2$ Seiberg-Witten theory. Our low energy quantum mechanics is consistent with this claim since

$$\begin{aligned} & \Omega(\Gamma + \gamma_h, \zeta) \mathcal{X}_{\gamma_h}(\zeta, R) \\ &= e^{-2\pi \text{Re}[\zeta^{-1} Z_{\gamma_h}]} \text{tr}_{\Gamma + \gamma_h} \left[(-1)^F e^{-2\pi R H_{\text{moduli}} - i\theta \cdot Q} \sigma(Q) \right] \times (\dots), \end{aligned} \quad (6.3)$$

where the first two terms follows from discussions in section 2, while $\sigma(Q)$ denotes the quadratic refinement, as argued in Ref. [20]. The trace is over the quantum mechanical Hilbert space for the charge $\Gamma + \gamma_h$, while the parenthesis denotes subleading loop contribution in the given charge sector.

An important generalization of our analysis is to study the wall-crossing phenomena in the $N = 2$ supergravity. In fact, the formalism we developed is more natural for the supergravity system, since the horizon provides natural cut-off at short distance and renders the Abelian description of the core state exact. That is, one can hide

the any potential subtlety associated with the Coulombic centers behind the horizon. Quantum mechanical description of more than one extremally charged black hole has been studied previously, but only in the context of same charge black holes, which is a particular limit of our dynamics without potential terms. We are poised to consider many black holes with mutually non-local and interacting center, and elevate Denef's old discussion black hole halos to fully quantum level.

In both field theory and the supergravity version of such a low energy quantum mechanics, there is a simpler way to count bound states. As long as the true moduli space defined by $\mathcal{K} = 0$ is compact, the relevant supercharge would be Fredholm, and one could compute the index by concentrating on the true moduli space defined by $\mathcal{K} = 0$. The quantum mechanics then would reduce to a supersymmetric Landau level problem on a curved $2n$ dimensional manifold, and can be presumably counted by computing the volume of this true moduli space. A similar idea has been recently used in [46, 47], but our approach provides a rigorous derivation of such a method and thus the precise state counting. Details of this computation will be presented elsewhere.

Finally, though we have focused on the moduli space dynamics of framed BPS particles in $D = 4$ $N = 2$ supersymmetric gauge theories, our analysis can be potentially applied to study the wall-crossing phenomena of any supersymmetric theories in presence of higher dimensional external objects. One potential application is a study of the wall-crossing formulae of the four-dimensional gauge theories in presence of a surface operator, which has been conjectured in [48] as a hybrid of 2D Ceccotti-Vafa WCF [3] and 4D Kontsevich-Soibelman WCF [19]. Our analysis also would be useful to study the wall-crossing formulae of two-dimensional $\mathcal{N} = (2, 2)$ massive \mathbb{CP}^n models in relation to that of four-dimensional $\mathcal{N} = 2$ SQCD [49, 50, 51, 52].

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Appendix

A BPS Equation for the Semiclassical Core

This appendix reviews the BPS equation, of Seiberg-Witten low energy theory, for long-range Abelian fields for any given core charges. One can easily read off $N = 2$ SUSY variation rules in four dimensions from $N = 1$ SUSY variation rules in six dimensions

$$\delta\lambda_A = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon_A, \quad (\text{A.1})$$

where λ and ϵ are six-dimensional chiral spinors,

$$\Gamma^{012345}\lambda_A = \lambda_A, \quad \Gamma^{012345}\epsilon_A = \epsilon_A. \quad (\text{A.2})$$

Here $A = 1, 2$ are the R-symmetry indices. Let us decompose the six-dimensional gamma matrices Γ^M as

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1}_2, \quad \Gamma^4 = \gamma_c \otimes \tau^2, \quad \Gamma^5 = \gamma_c \otimes \tau^1, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (\text{A.3})$$

where $i\gamma_c = \gamma^{0123}$. In the above representation, the gaugino λ_A can be decomposed into $\lambda_A = \lambda_{\alpha A} \oplus \bar{\lambda}_{\dot{\alpha} A}$. As usual, $\alpha, \dot{\alpha}$ denote the 4-D chiral/anti-chiral spinor indices. One can then rewrite (A.1) as

$$\delta\lambda_{\alpha A} = \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\epsilon_{\beta A} + i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\epsilon}_{\dot{\alpha}}D_\mu\phi, \quad \phi = A_4 + iA_5. \quad (\text{A.4})$$

With $Z_c = |Z_c|\zeta$, the core state configuration should satisfy the following relation

$$\left[(Q^A + i\zeta^{-1}\bar{Q}^A\bar{\sigma}^0)\epsilon_A, \lambda_B \right] = 0 \quad (\text{A.5})$$

or equivalently

$$-i\vec{\tau}\epsilon_B \cdot (\vec{B} + i\vec{E} - i\zeta^{-1}\vec{\nabla}\phi) - \zeta^{-1}\epsilon_B\partial_t\phi = 0, \quad (\text{A.6})$$

that is,

$$\vec{\mathcal{F}} - i\zeta^{-1}\vec{\nabla}\phi = 0, \quad \partial_t\phi = 0. \quad (\text{A.7})$$

One quick way to show that ζ represents the phase factor of Z_c is to look at the energy for the configuration (A.7), say, for rank one example: performing the usual trick of completing the square with (A.7) in mind, one obtain

$$\mathcal{E} = \frac{1}{8\pi} \int d^3\mathbf{x} \operatorname{Im}\tau \left[\vec{B}^2 + \vec{E}^2 + |\vec{\nabla}\phi|^2 \right] = \operatorname{Re} \left[\zeta^{-1}Z_c \right], \quad (\text{A.8})$$

with $Z_c = P\phi_D(\infty) + Q\phi(\infty)$. This shows that $\zeta^{-1}Z_c = |Z_c|$.

B More on $\mathcal{N} = 4$ Quantum Mechanics

Here we present more on $\mathcal{N} = 4$ Lagrangian with conformal R^3 target manifold. Here, we first derive the massless case with curved background and then add potential terms, which provides an alternate path to (3.28). Then, we spend some time on supercharge operators and quantum Hamiltonian.

B.1 Massless and curved

First of all, we wish to fill the gap between sections 3.1 and 3.3 with a derivation of massless $\mathcal{N} = 4$ theory onto conformally flat R^3 , which turned out to be regarded as a special case of theories in Ref. [22]. In next subsection, we demonstrate that how the massive Lagrangian of section 3.3 emerges by combining the result of section 3.1 with this massless case. Based on the educated guess and group theoretical consideration, one possible candidate for $\mathcal{N} = 4$ SUSY transformation rules are following

$$\delta x^a = i\eta_{mn}^a \epsilon^m \psi^n, \quad \delta \psi^m = \eta_{mn}^a \epsilon^n \dot{x}^a + \alpha \epsilon_m \eta_{pq}^a f^{-1} \partial_a f \psi^p \psi^q, \quad (\text{B.1})$$

where α will be determined. Here η_{mn}^a denotes the 't Hooft tensor with the convention $\eta_{12}^3 = \eta_{34}^1 = +1$.

To start, consider a standard kinetic term for flat target manifold,

$$\mathcal{L}^{(0)} = \frac{1}{2} f \dot{x}^a \dot{x}^a + \frac{i}{2} f \psi^m \dot{\psi}^m, \quad (\text{B.2})$$

whose variation under the $\mathcal{N} = 4$ SUSY transformations is

$$\begin{aligned} \delta\left(\frac{1}{2} f \dot{x}^a \dot{x}^a\right) &= \frac{i}{2} \eta_{mn}^b \partial_b f \epsilon^m \psi^n \dot{x}^a \dot{x}^a + i f \eta_{mn}^a \epsilon^m \dot{\psi}^n \dot{x}^a, \\ \delta\left(\frac{i}{2} f \psi^m \dot{\psi}^m\right) &= -\frac{1}{2} \eta_{pq}^a \partial_a f \epsilon^p \psi^q \psi^m \dot{\psi}^m - i f \eta_{mn}^a \epsilon^m \dot{\psi}^n \dot{x}^a - \frac{i}{2} \eta_{mn}^a \partial_b f \dot{x}^a \dot{x}^b \epsilon^m \psi^n \\ &\quad + i \alpha \eta_{mn}^a \partial_a f \psi^m \psi^n \epsilon^p \dot{\psi}^p + \frac{i}{2} \alpha \eta_{mn}^a f^{-1} \partial_a f \partial_l f \dot{x}^l \psi^m \psi^n \epsilon^p \psi^p. \end{aligned} \quad (\text{B.3})$$

- One can reorganize the velocity-square terms in (B.3) into

$$\begin{aligned} \frac{i}{2} \partial_b f \epsilon^m \psi^n \left[\eta_{mn}^b \dot{x}^a - \eta_{mn}^a \dot{x}^b \right] \dot{x}^a &= \frac{i}{2} \epsilon_{eab} \epsilon_{ecd} \dot{x}^a \dot{x}^c \partial_b f \eta_{mn}^d \epsilon^m \psi^n \\ &= + \frac{i}{2} \eta_{pm}^c \eta_{np}^e \epsilon_{eab} \dot{x}^a \dot{x}^c \partial_b f \epsilon^m \psi^n \\ &= - \frac{i}{2} \epsilon_{eab} \cdot \dot{x}^a \partial_b f \eta_{np}^e \delta \psi^n \psi^p - \frac{i}{2} \alpha \dot{x}^a f^{-1} \partial_a f \partial_b f \eta_{mn}^b \psi^m \psi^n \epsilon^p \psi^p \\ &\quad + \frac{i}{6} \alpha f^{-1} \partial_a f \partial_a f \epsilon_{mnpq} \psi^m \psi^n \psi^p \delta \psi^q. \end{aligned} \quad (\text{B.4})$$

- The first term in the last equality of (B.4) implies that we have to add the following term

$$\begin{aligned} \delta\left(+\frac{i}{4}\epsilon_{abc}\dot{x}^a\partial_b f\eta_{mn}^c\psi^m\psi^n\right) &= +\frac{i}{2}\epsilon_{abc}\dot{x}^a\partial_b f\eta_{mn}^c\delta\psi^m\psi^n \\ &\quad -\frac{1}{4}\epsilon_{abc}\eta_{pq}^a\eta_{mn}^c\partial_b f\epsilon^p\dot{\psi}^q\psi^m\psi^n -\frac{1}{4}\epsilon_{abc}\dot{x}^a\partial_b\partial_d f\eta_{mn}^c\eta_{pq}^d\epsilon^p\psi^q\psi^m\psi^n . \end{aligned} \quad (\text{B.5})$$

- Using the identities of 't Hooft tensor

$$\begin{aligned} \epsilon_{abc}\eta_{mn}^c\eta_{pq}^a &= \delta_{mp}\eta_{nq}^b - \delta_{np}\eta_{mq}^b + \delta_{nq}\eta_{mp}^b - \delta_{mq}\eta_{np}^b , \\ \eta_{pq}^d\eta_{mn}^c + \eta_{pm}^d\eta_{nq}^c + \eta_{pn}^d\eta_{qm}^c + \eta_{ps}^d\eta_{rs}^c\epsilon_{qmnr} &= 0 , \end{aligned} \quad (\text{B.6})$$

one can massage the second and third terms in (B.5) into followings:

$$-\frac{1}{4}\epsilon_{abc}\eta_{pq}^a\eta_{mn}^c\partial_b f\epsilon^p\dot{\psi}^q\psi^m\psi^n = \frac{1}{2}\eta_{mn}^a\partial_a f\epsilon^m\psi^n \cdot \psi^p\dot{\psi}^p - \frac{1}{2}\eta_{mn}^a\partial_a f\psi^m\dot{\psi}^n \cdot \epsilon^p\psi^p , \quad (\text{B.7})$$

and

$$\begin{aligned} -\frac{1}{4}\epsilon_{abc}\dot{x}^a\partial_b\partial_d f\eta_{mn}^c\eta_{pq}^d\epsilon^p\psi^q\psi^m\psi^n &= +\frac{1}{12}\dot{x}^a\partial_b\partial_d f\eta_{ps}^d\epsilon_{abc}\eta_{rs}^c\epsilon_{qmnr}\epsilon^p\psi^q\psi^m\psi^n \\ &= +\frac{1}{12}\epsilon_{mnpq}\partial^2 f\psi^m\psi^n\psi^p\delta\psi^q \\ &\quad -\frac{1}{12}\dot{x}^a\partial_a\partial_c f\eta_{pl}^c\epsilon^p\psi^q\psi^m\psi^n\epsilon_{qmnl} . \end{aligned} \quad (\text{B.8})$$

In summary, one can show that

$$\begin{aligned} \delta\left(+\frac{i}{4}\epsilon_{abc}\dot{x}^a\partial_b f\eta_{mn}^c\psi^m\psi^n\right) &= +\frac{i}{2}\epsilon_{abc}\dot{x}^a\partial_b f\eta_{mn}^c\delta\psi^m\psi^n + \frac{1}{2}\eta_{mn}^a\partial_a f\epsilon^m\psi^n \cdot \psi^p\dot{\psi}^p \\ &\quad + \frac{1}{4}\eta_{mn}^a\partial_a f\psi^m\dot{\psi}^n \cdot \epsilon^p\dot{\psi}^p + \frac{1}{12}\epsilon_{mnpq}\partial^2 f\psi^m\psi^n\psi^p\delta\psi^q . \end{aligned} \quad (\text{B.9})$$

- Here one can determine, from the fourth term in second equality of (B.3) and third term in (B.9), the value of the coefficient α by

$$\alpha = +\frac{i}{4} \quad (\text{B.10})$$

- Collecting all the results so far, one can have

$$\begin{aligned} & \delta \left(\frac{1}{2} f \dot{x}^a \dot{x}^a + \frac{i}{2} f \psi^m \dot{\psi}^m + \frac{i}{4} \epsilon_{abc} \dot{x}^a \partial_b f \eta_{mn}^c \psi^m \psi^n \right) \\ &= \frac{1}{12} \epsilon_{mnpq} \partial^2 f \psi^m \psi^n \psi^p \psi^q \delta \psi^q - \frac{1}{24} f^{-1} \partial_a f \partial_a f \epsilon_{mnpq} \psi^m \psi^n \psi^p \delta \psi^q . \end{aligned} \quad (\text{B.11})$$

At the end of the day, this gives the massless $\mathcal{N} = 4$ non-linear sigma model therefore takes the following form

$$\begin{aligned} \mathcal{L}^{(0)} &= \frac{1}{2} f \dot{x}^a \dot{x}^a + \frac{i}{2} f \psi^m \dot{\psi}^m + \frac{i}{4} \epsilon_{abc} \dot{x}^a \partial_b f \eta_{mn}^c \psi^m \psi^n \\ &\quad - \frac{1}{48} \partial_a^2 f \epsilon_{mnpq} \psi^m \psi^n \psi^p \psi^q + \frac{1}{96} f^{-1} (\partial_a f)^2 \epsilon_{mnpq} \psi^m \psi^n \psi^p \psi^q , \end{aligned} \quad (\text{B.12})$$

where the covariant derivative for fermions is defined as

$$\nabla_t \psi^m = \dot{\psi}^m + \frac{1}{2} \epsilon_{abc} \dot{x}^a \partial_b \log f \eta_{mn}^c \psi^n . \quad (\text{B.13})$$

The above massless Lagrangian is invariant under the $\mathcal{N} = 4$ SUSY transformation

$$\delta x^a = i \eta_{mn}^a \epsilon^m \psi^n , \quad \delta \psi_m = \eta_{mn}^a \epsilon^n \dot{x}^a + \frac{i}{4} \epsilon_m \eta_{pq}^a f^{-1} \partial_a f \psi^p \psi^q . \quad (\text{B.14})$$

This is the curved space version of (3.2).

B.2 Massive and curved

Now we wish to add potential terms to this by twisting the supersymmetry transformation rules. From discussion of section 3.2, it is clear that the right thing to do, at least in the context of $\mathcal{N} = 1$ supersymmetry, is to shift the fermion transformation rule as

$$\delta x^a = i \eta_{mn}^a \epsilon^m \psi^n , \quad \delta \psi_m = \eta_{mn}^a \epsilon^n \dot{x}^a + \epsilon_m \frac{1}{f} \left(\mathcal{K} + \frac{i}{4} \eta_{pq}^a \partial_a f \psi^p \psi^q \right) , \quad (\text{B.15})$$

since the last piece multiplying ϵ_m is nothing but the on-shell value of the auxiliary field b . The corresponding Lagrangian from (3.28)

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} f \left[\dot{x}^a \dot{x}^a + i \psi^m \nabla_t \psi^m - \frac{1}{4!} \epsilon_{mnpq} \{ \nabla^2 f - (\partial_a \log f)^2 \} \psi^m \psi^n \psi^p \psi^q \right] \\ &\quad - \mathcal{W}_a \dot{x}^a + i \partial_b \mathcal{W}_c \psi^b \psi^c + i f^{1/2} \partial_a (f^{-1/2} \mathcal{K}) \psi^a \lambda - \frac{i}{4} \epsilon_{abc} \mathcal{K} f^{-1} \partial_a f \psi^b \psi^c - \frac{1}{2f} \mathcal{K}^2 \end{aligned} \quad (\text{B.16})$$

is indeed consistent with the above massless one in (B.12).

To show that this Lagrangian is invariant under this transformation, we split it into three parts, $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)}$, as

$$\begin{aligned}
\mathcal{L}^{(0)} &= \frac{1}{2}f\dot{x}^a\dot{x}^a + \frac{i}{2}f\psi^m\dot{\psi}^m + \frac{i}{4}\epsilon_{abc}\dot{x}^a\partial_b f\eta_{mn}^c\psi^m\psi^n - \mathcal{W}_a\dot{x}^a \\
&\quad - \frac{1}{48}\partial_a^2 f\epsilon_{mnpq}\psi^m\psi^n\psi^p\psi^q + \frac{1}{96}f^{-1}(\partial_a f)^2\epsilon_{mnpq}\psi^m\psi^n\psi^p\psi^q, \\
\mathcal{L}^{(1)} &= \frac{i}{2}f^{1/2}\partial_a(f^{1/2}K)\eta_{mn}^a\psi^m\psi^n, \\
\mathcal{L}^{(2)} &= -\frac{1}{2}fK^2,
\end{aligned} \tag{B.17}$$

where we introduced $K \equiv f^{-1}\mathcal{K}$. $\mathcal{L}^{(0)}$ is already invariant under (B.14), so we have only K -dependence pieces in $\delta\mathcal{L}^{(0)}$, which is

$$\begin{aligned}
\delta\mathcal{L}^{(0)} &= -if^{1/2}\partial_a(f^{1/2}K)\dot{x}^a\epsilon^m\psi_m - i\epsilon_{abc}\dot{x}^af^{1/2}\partial_b(f^{1/2}K)\eta_{mn}^ce_me_m\psi_n \\
&\quad - \frac{1}{12}K\left[\partial_a^2 f - \frac{1}{2}f^{-1}(\partial_a f)^2\right]\epsilon_{mnpq}\psi^m\psi^n\psi^p\psi^q.
\end{aligned} \tag{B.18}$$

After some tedious computation, we find

$$\begin{aligned}
\delta\mathcal{L}^{(1)} &= \frac{i}{2}\delta\left(f^{1/2}\partial_a(f^{-1/2}\cdot fK)\right)\eta_{mn}^a\psi^m\psi^n + if^{1/2}\partial_a(f^{1/2}K)\delta\psi^m\psi^n \\
&= \frac{1}{12}K\partial_a^2 f\epsilon_{mnpq}\psi^m\psi^n\psi^p\psi^q - \frac{1}{24}Kf^{-1}(\partial_a f)^2\epsilon_{mnpq}\psi^m\psi^n\psi^p\psi^q \\
&\quad + if^{1/2}\partial_a(f^{1/2}K)\dot{x}^a\epsilon^m\psi_m + i\epsilon_{abc}\dot{x}^af^{1/2}\partial_b(f^{1/2}K)\eta_{mn}^ce_me_m\psi_n \\
&\quad + \delta\left(\frac{1}{2}fK^2\right),
\end{aligned} \tag{B.19}$$

which, combined with $\delta\mathcal{L}^{(0)}$, give us

$$\delta(\mathcal{L}^{(0)} + \mathcal{L}^{(1)}) = \delta\left(\frac{1}{2}fK^2\right) = -\delta\mathcal{L}^{(2)}, \tag{B.20}$$

as required.

B.3 Supercharges and Hamiltonian

Using the $\mathcal{N} = 4$ supersymmetric variation rules (3.30), the Nöther charges of the $\mathcal{N} = 4$ supersymmetry therefore become

$$Q_m = -\eta_{mn}^a\psi^n(p_a + \mathcal{W}_a) + \frac{i}{4}\eta_{mn}^af^{-1}\partial_af\psi^n + \frac{i}{4}\partial_af\eta_{pq}^a\psi^{[p}\psi^q\psi^m] + \mathcal{K}\psi^m. \tag{B.21}$$

For completeness, let us check whether the above supercharges give the correct supersymmetric transformation rules for bosons and fermions. One can read off from the Lagrangian (3.28) the canonical quantization

$$[x^a, p_b] = i\delta_b^a, \quad \{\psi^m, \psi^n\} = f^{-1}\delta^{mn}, \quad [p_a, \psi^m] = \frac{i}{2}f^{-1}\partial_a f \psi^m. \quad (\text{B.22})$$

One can show

$$\begin{aligned} & \{ -\eta_{mp}^a \psi^p (p_a + \mathcal{W}_a), \psi^n \} \\ &= -\eta_{mn}^a f^{-1} (p_a + \mathcal{W}_a) - \frac{i}{2} \eta_{mp}^a f^{-1} \partial_a f \psi^p \psi^n \\ &= -\eta_{mn}^a \dot{x}^a - \frac{i}{4} \eta_{mp}^a f^{-1} \partial_a f \{ \psi^p, \psi^n \} + \frac{i}{2} f^{-1} \partial_a f \eta_{np}^a \psi^{[m} \psi^{p]} , \\ &= -\eta_{mn}^a \dot{x}^a + \frac{i}{2} f^{-1} \partial_a f \eta_{np}^a \psi^{[m} \psi^{p]} - \frac{i}{4} f^{-2} \partial_a f \eta_{mn}^a , \end{aligned} \quad (\text{B.23})$$

where we used for the second equality the definition of momentum operator p_a

$$p_a + \mathcal{W}_a = f \dot{x}^a + \frac{i}{4} \epsilon_{abc} \partial_b f \eta_{mn}^c \psi^m \psi^n. \quad (\text{B.24})$$

One can also show that

$$\begin{aligned} & \{ \frac{i}{4} \partial_a f \eta_{pq}^a \psi^{[p} \psi^q \psi^m], \psi^n \} \\ &= \delta_{mn} \frac{i}{4} f^{-1} \partial_a f \eta_{pq}^a \psi^p \psi^q + \frac{i}{2} f^{-1} \partial_a f \eta_{np}^a \psi^p \psi^m + \frac{i}{4} f^{-2} \partial_a f \eta_{mn}^a , \end{aligned} \quad (\text{B.25})$$

where we used an identity of 't Hooft tensor

$$\epsilon_{abc} \eta_{mn}^b \eta_{pq}^c = \eta_{mp}^a \delta_{nq} - \eta_{np}^a \delta_{mq} + \eta_{nq}^a \delta_{mp} - \eta_{mq}^a \delta_{np}. \quad (\text{B.26})$$

It implies that

$$\{Q_m, \psi_n\} = -\eta_{mn}^a \dot{x}^a + \delta_{mn} f^{-1} \left(\mathcal{K} + \frac{i}{4} f^{-1} \partial_a f \eta_{pq}^a \psi^p \psi^q \right), \quad (\text{B.27})$$

while the action of supercharges on the bosons follows immediately,

$$[Q_m, x^a] = i \eta_{mn}^a \psi^n. \quad (\text{B.28})$$

These are precisely the supersymmetry transformation rules in (3.30).

Finally, we wish to determine the quantum form of the Hamiltonian using

$$Q_4^2 = H.$$

Let us first write

$$Q_4 = \psi^a(p + \mathcal{W})_a + \lambda(\mathcal{K} + Z) ,$$

where

$$Z = \frac{i}{2} \partial_a f \psi^a \lambda + \frac{i}{4} \epsilon_{abc} \partial_a f \psi^b \psi^c .$$

Using $\{Q_4, \lambda\} = (\mathcal{K} + Z)/f$ and $\{Q_4, \psi^a\} = \dot{x}^a = f^{-1} \pi_a$, with the supercovariant momentum operator

$$\pi_a = (p + \mathcal{W})_a + \Gamma_a, \quad \Gamma_a \equiv \frac{i}{2} \partial_b f \psi^{[b} \psi^{a]} - \frac{i}{2} \epsilon_{abc} \partial_b f \psi^c \lambda ,$$

we find

$$\begin{aligned} \{Q_4, Q_4\} &= \{Q_4, \psi^a(p + \mathcal{W})_a + \lambda(\mathcal{K} + Z)\} \\ &= \frac{1}{f} \pi^a (p + \mathcal{W})_a + \frac{1}{f} (\mathcal{K} + Z)^2 \\ &\quad - \psi^a [Q_4, (p + \mathcal{W})_a] - \lambda [Q_4, \mathcal{K} + Z] . \end{aligned} \quad (\text{B.29})$$

Let us separate out terms involving either \mathcal{W} or \mathcal{K} from the last two terms. Using $d\mathcal{K} = *d\mathcal{W}$, we find

$$\begin{aligned} \{Q_4, Q_4\} &= \frac{1}{f} \pi^a (p + \mathcal{W})_a + \frac{1}{f} (\mathcal{K} + Z)^2 - 2i \partial_a \mathcal{K} \psi^a \lambda - i \epsilon_{abc} \partial_a \mathcal{K} \psi^b \psi^c \\ &\quad + (\psi^a [(p + \mathcal{W})_a, \psi^b] + \lambda [Z, \psi^b]) (p + \mathcal{W})_b \\ &\quad + (\psi^a [(p + \mathcal{W})_a, \lambda] + \lambda [Z, \lambda]) Z \\ &\quad + 2\psi^a \lambda [(p + \mathcal{W})_a, Z] . \end{aligned} \quad (\text{B.30})$$

By explicit computation one can see that

$$\begin{aligned} (\psi^a [(p + \mathcal{W})_a, \psi^b] + \lambda [Z, \psi^b]) &= \frac{1}{f} \Gamma_b \\ (\psi^a [(p + \mathcal{W})_a, \lambda] + \lambda [Z, \lambda]) &= 0 . \end{aligned} \quad (\text{B.31})$$

Since $[(p + \mathcal{W})_a, \Gamma_a] = 0$ upon the summation over a ,

$$\frac{1}{f} \pi^a (p + \mathcal{W})_a + \frac{1}{f} \Gamma_b (p + \mathcal{W})_b = \frac{1}{f} \pi_a \pi_a - \frac{1}{f} \Gamma_a \Gamma_a . \quad (\text{B.32})$$

Finally expanding $(\mathcal{K} + Z)^2$ out, we complete the potential terms associated with \mathcal{K} from $\mathcal{K}^2 + 2\mathcal{K}Z$, but have a leftover piece Z^2 . So combining them all, we have

$$\begin{aligned} \{Q_4, Q_4\} &= \frac{1}{f}\pi^a\pi_a + \frac{1}{f}\mathcal{K}^2 - 2if^{1/2}\partial_a(f^{-1/2}\mathcal{K})\psi^a\lambda - i\epsilon_{abc}f^{1/2}\partial_a(f^{-1/2}\mathcal{K})\psi^b\psi^c \\ &\quad + \frac{1}{f}Z^2 - \frac{1}{f}\Gamma_b\Gamma_b + 2\psi^a\lambda[(p + \mathcal{W})_a, Z] \end{aligned} \quad (\text{B.33})$$

The last line can be organized in terms of the curvature of the fermion bundle,

$$[D_a, D_b] = F_{abmn}\psi^m\psi^n, \quad D_a \equiv \partial_a + i\Gamma_a, \quad (\text{B.34})$$

and has the explicit form,

$$\begin{aligned} -\frac{1}{2}F_{abmn}\psi^a\psi^b\psi^m\psi^n &= \frac{1}{48} \left(2(\partial^2 f) - f^{-1}(\partial f)^2 \right) \epsilon_{mnkl}\psi^m\psi^n\psi^k\psi^l \\ &\quad - \frac{1}{4}f^{-2}(\partial^2 f) + \frac{1}{8}f^{-3}(\partial f)^2, \end{aligned} \quad (\text{B.35})$$

Thus, the Hamiltonian $H = \{Q_4, Q_4\}/2$ is

$$H = \frac{1}{2f}\pi_a\pi_a - \frac{1}{4}F_{abmn}\psi^a\psi^b\psi^m\psi^n + \frac{1}{2f}\mathcal{K}^2 - \frac{i}{2}\eta_{mn}^a f^{1/2}\partial_a(f^{-1/2}\mathcal{K})\psi^m\psi^n \quad (\text{B.36})$$

Although $SU(2)_R$ is not manifest in the curvature piece, it is actually $SU(2)_R$ invariant as can be seen from (B.35). This coincides with the classical Hamiltonian up to normal ordering; the curvature pieces generate extra terms because quantum ψ 's obey not the Grassman algebra but the Clifford algebra.

Note that the kinetic term is slightly unconventional in its choice of normal ordering. Because of this, the inner product in the Hilbert space of this quantum mechanics should be defined as

$$||\Psi||^2 = \int dx^3 f \Psi^\dagger \Psi. \quad (\text{B.37})$$

More usual choice of kinetic term/inner product is related to our convention by rescaling of the wavefunction by a factor of $f^{1/4}$.

C Review of KS Invariant and Line Operator

The idea of the framed BPS state originally arises in study of four-dimensional $N = 2$ supersymmetric theories in presence of an external particle of charge Γ , called line operator L_Γ . The line operator can be characterized by the phase factor ζ of its

central charge Z_Γ . Compactifying the theory on a circle, it has been conjectured in [20] that the vacuum expectation value of L_Γ can be expanded in terms of the Darboux coordinates \mathcal{X}_γ with integer coefficients

$$\langle L_\Gamma \rangle = \sum_{\gamma} \Omega(\Gamma + \gamma) \mathcal{X}_\gamma , \quad (\text{C.1})$$

which provides us a direct physical interpretation of Darboux coordinates. Each integer coefficient $\Omega(\Gamma + \gamma)$ here represents the supersymmetric index of a framed BPS state of charge γ bounded to L_Γ . The Darboux coordinates are very useful to compute the hyperKähler metric on the Coulomb branch of four-dimensional theories on a circle.

The expectation value of the line operator depends on both ζ and the Coulomb branch parameter a in four-dimensional theories. Due to the fact that the physical observable $\langle L_\Gamma \rangle$ should not have any discontinuities as ζ and a change, important consequences of (C.1) are that one can understand how the Kontsevich-Soibelman invariant naturally arises, and that provides the origin of the thermodynamic Bethe ansatz equation the Darboux coordinates should satisfy.

Let us now review in this section the central importance of semi-primitive wall-crossing formula to derive the Kontsevich-Soibelman BPS invariant in the context of line operators. For more details, it is referred to [20].

As discussed in the main context, the Witten index $\Omega(\Gamma + \gamma, \zeta)$ can jump once the phase of central charge for a certain probe(halo) particle of γ_h is parallel to that of the external particle of Γ denoted by $\arg(\zeta)$. That is, when ζ moves across the so-called BPS ray $l_h = \{\zeta \mid Z_h/\zeta \in R_+\}$, the index could have discontinuity. One advantage on computation of the index jump in presence of line operator is that the wall-crossing phenomena is essentially restricted to the semi-primitive ones.

Let us now consider the vacuum expectation value of the line operator conjectured as in (C.1)

$$\langle L_\Gamma \rangle = \sum_{\gamma} \Omega(\Gamma + \gamma) \mathcal{X}_\gamma ,$$

where \mathcal{X}_γ satisfy a multiplication rule below

$$\mathcal{X}_{\gamma_1} \mathcal{X}_{\gamma_2} = (-1)^{2\langle \gamma_1, \gamma_2 \rangle} \mathcal{X}_{\gamma_1 + \gamma_2} . \quad (\text{C.2})$$

Let us then increase the phase parameter $\arg(\zeta)$ so that it moves across the BPS ray l_h .

- Look at the relation (2.22). If $\langle \gamma_c, \gamma_h \rangle > 0$, we have a stable bound state between core and halo particles before ζ cross the BPS ray l_h . Then, one can reorganize

(C.2) before across the ray into the following form

$$\begin{aligned}\langle L_\Gamma \rangle_- &= \sum_{\gamma_c} \mathcal{X}_{\gamma_c} \cdot \sum_{n=0} \Omega(\gamma_c + n\gamma_h) (-1)^{2n\langle \gamma_c, \gamma_h \rangle} \mathcal{X}_{\gamma_h}^n, \\ &= \sum_{\gamma_c} \Omega(\gamma_c) \mathcal{X}_{\gamma_c} \prod_{n=1} \left[1 - \mathcal{X}_{\gamma_h}^n \right]^{2n\langle \gamma_c, \gamma_h \rangle \Omega(n\gamma_h)}.\end{aligned}\quad (\text{C.3})$$

Note that we used the semi-primitive wall-crossing formula (5.17) for the last equality. Since we loose the Fock space of halo particles after across the ray l_h , one can say that

$$\langle L_\Gamma \rangle_+ = \sum_{\gamma_c} \Omega_{\gamma_c} \mathcal{X}_{\gamma_c} . \quad (\text{C.4})$$

One can therefore conclude that, since $\langle L_\Gamma \rangle$ should be continuous across the ray, \mathcal{X}_{γ_c} is required to jump across the wall by the amount

$$\mathcal{X}_{\gamma_c} \rightarrow \mathcal{X}_{\gamma_c} \prod_{n=1} \left[1 - \mathcal{X}_{\gamma_h}^n \right]^{2n\langle \gamma_h, \gamma_c \rangle \Omega(n\gamma_h)} = \prod_{n=1} \mathcal{K}_{n\gamma_h}^{\Omega(n\gamma_h)}(\mathcal{X}_{\gamma_c}) , \quad (\text{C.5})$$

where

$$\mathcal{K}_{\gamma_h}(\mathcal{X}_{\gamma_c}) = \mathcal{X}_{\gamma_c} \left[1 - \mathcal{X}_{\gamma_h} \right]^{2\langle \gamma_h, \gamma_c \rangle} . \quad (\text{C.6})$$

It is noteworthy here that this is the desired discontinuity how the Darboux coordinate \mathcal{X}_γ jumps across the BPS ray l_h .

- Let us now in turn consider the converse, i.e., $\langle \gamma_c, \gamma_h \rangle < 0$. According to (2.22), there is no stable bound state between the core and halo particle before the ζ across the BPS ray l_h . Then, one can rewrite (C.2) before across the ray as

$$\langle L_\Gamma \rangle_- = \sum_{\gamma_c} \Omega_{\gamma_c} \mathcal{X}_{\gamma_c} . \quad (\text{C.7})$$

Since we gain the Fock space of halo particles after across the ray l_h , one can say that

$$\langle L_\Gamma \rangle_+ = \sum_{\gamma_c} \Omega(\gamma_c) \mathcal{X}_{\gamma_c} \prod_{n=1} \left[1 - \mathcal{X}_{\gamma_h}^n \right]^{-2n\langle \gamma_c, \gamma_h \rangle \Omega(n\gamma_h)} . \quad (\text{C.8})$$

\mathcal{X}_{γ_c} is again required to jump across the wall by the same amount

$$\mathcal{X}_{\gamma_c} \rightarrow \mathcal{X}_{\gamma_c} \prod_{n=1} \left[1 - \mathcal{X}_{\gamma_h}^n \right]^{2n\langle \gamma_h, \gamma_c \rangle \Omega(n\gamma_h)} = \prod_{n=1} \mathcal{K}_{n\gamma_h}^{\Omega(n\gamma_h)}(\mathcal{X}_{\gamma_c}) , \quad (\text{C.9})$$

which is the same to (C.6).

Let us now consider two chambers of $\mathcal{M}_{\text{Coulomb}} \times \mathbb{C}^*$, the Coulomb branch and ζ -plane, separated by walls of marginal stability. The physical observable $\langle L_{\Gamma} \rangle$ should not depend on choice of a path connecting those two chambers. The different paths however in general cross different set of walls of marginal stability. One can therefore conclude, from the fact that there are infinitely many possible line operators, that a path-ordered product of transformations below

$$\mathcal{I} = \prod_{\gamma_h}^{\curvearrowright} \prod_n \mathcal{K}_{n\gamma_h}^{\Omega(n\gamma_h)} \quad (\text{C.10})$$

defines an invariant over the Coulomb branch $\mathcal{M}_{\text{Coulomb}}$. \mathcal{I} is indeed the so-called Kontsevich-Soibelman invariant.

References

- [1] M.K. Prasad and C.M. Sommerfield, “*An Exact Classical Solution for the ’t Hooft Monopole and the Julia-Zee Dyon*,” Phys. Rev. Lett. **35** (1975) 760.
- [2] E.B. Bogomolny, “*Stability of Classical Solutions*,” Sov. J. Nucl. Phys. **24** (1976) 449 [Yad. Fiz. **24** (1976) 861].
- [3] S. Cecotti and C. Vafa, “*On Classification of $\mathcal{N} = 2$ Supersymmetric Theories*,” Commun. Math. Phys. **158** (1993) 569 [arXiv:hep-th/9211097].
- [4] S. Cecotti, P. Fendley, K.A. Intriligator and C. Vafa, “*A New Supersymmetric Index*,” Nucl. Phys. B **386** (1992) 405 [arXiv:hep-th/9204102].
- [5] N. Seiberg and E. Witten, “*Monopole Condensation, And Confinement In $N=2$ Supersymmetric Yang-Mills Theory*,” Nucl. Phys. B **426** (1994) 19 [Erratum-ibid. B **430** (1994) 485] [arXiv:hep-th/9407087].
- [6] N. Seiberg and E. Witten, “*Monopoles, Duality and Chiral Symmetry Breaking in $N=2$ Supersymmetric QCD*,” Nucl. Phys. B **431** (1994) 484 [arXiv:hep-th/9408099].
- [7] F. Ferrari and A. Bilal, “*The Strong-Coupling Spectrum of the Seiberg-Witten Theory*,” Nucl. Phys. B **469**, 387 (1996) [arXiv:hep-th/9602082].
- [8] K.M. Lee and P. Yi, “*Dyons in $N=4$ Supersymmetric Theories and Three Pronged Strings*,” Phys. Rev. **D58** (1998) 066005. [hep-th/9804174].
- [9] D. Bak, C.K. Lee, K.M. Lee, and P. Yi “*Low-energy Dynamics for $1/4$ BPS Dyons*,” Phys. Rev. **D61** (2000) 025001. [hep-th/9906119].

- [10] J.P. Gauntlett, N. Kim, J. Park and P. Yi “*Monopole Dynamics and BPS Dyons N=2 Super Yang-Mills Theories*,” Phys. Rev. **D61** (2000) 125012. [hep-th/9912082].
- [11] D. Bak, K. -M. Lee and P. Yi, “Complete supersymmetric quantum mechanics of magnetic monopoles in N=4 SYM theory,” Phys. Rev. **D62** (2000) 025009. [hep-th/9912083].
- [12] J.P. Gauntlett, C.J. Kim, K.M. Lee and P.Yi “*General Low-energy Dynamics of Supersymmetric Monopoles*,” Phys. Rev. **D63** (2001) 065020. [hep-th/0008031].
- [13] D. Bak, K.M. Lee and P. Yi, “*Quantum 1/4 BPS Dyons*,” Phys. Rev. **D61** (2000) 045003. [hep-th/9907090].
- [14] M. Stern and P. Yi, “*Counting Yang-Mills Dyons with Index Theorems*,” Phys. Rev. **D62** (2000) 125006. [hep-th/0005275].
- [15] F. Denef, “*Supergravity Flows and D-brane Stability*,” JHEP **0008** (2000) 050 [arXiv:hep-th/0005049].
- [16] F. Denef, B.R. Greene and M. Raugas, “*Split Attractor Flows and the Spectrum of BPS D-branes on the Quintic*,” JHEP **0105** (2001) 012 [arXiv:hep-th/0101135].
- [17] F. Denef, “*Quantum Quivers and Hall/Hole Halos*,” JHEP **0210** (2002) 023 [arXiv:hep-th/0206072].
- [18] F. Denef and G.W. Moore, “*Split States, Entropy enigmas, Holes and Halos*,” arXiv:hep-th/0702146.
- [19] M. Kontsevich and Y. Soibelman, “*Stability Structures, Motivic Donaldson-Thomas Invariants and Cluster Transformations*,” arXiv:0811.2435
- [20] D. Gaiotto, G.W. Moore and A. Neitzke, “*Framed BPS States*,” arXiv:1006.0146 [hep-th].
- [21] R.A. Coles and G. Papadopoulos, “*The Geometry of the One-Dimensional Supersymmetric Nonlinear Sigma Models*,” Class. Quant. Grav. **7** (1990) 427.
- [22] A. Maloney, M. Spradlin and A. Strominger, “*Superconformal Multi-Black Hole Moduli Spaces in Four Dimensions*,” JHEP **0204** (2002) 003 [arXiv:hep-th/9911001].

- [23] A. Mikhailov, N. Nekrasov and S. Sethi, “Geometric realizations of BPS states in $N = 2$ theories,” Nucl. Phys. B **531** (1998) 345 [arXiv:hep-th/9803142].
- [24] P.C. Argyres and K. Narayan, “*String Webs from Field Theory*,” JHEP **0103** (2001) 047 [arXiv:hep-th/0101114].
- [25] A. Ritz, M.A. Shifman, A.I. Vainshtein and M.B. Voloshin, “*Marginal Stability and the Metamorphosis of BPS States*,” Phys. Rev. D **63** (2001) 065018 [arXiv:hep-th/0006028].
- [26] O. Bergman, “*Three Pronged Strings and $1/4$ BPS States in $N=4$ Super Yang-Mills Theory*,” Nucl. Phys. B **525** (1998) 104-116. [hep-th/9712211].
- [27] K. Lee, E.J. Weinberg and P. Yi, “*The Moduli Space of Many BPS Monopoles for Arbitrary Gauge Groups*,” Phys. Rev. D **54** (1996) 1633 [arXiv:hep-th/9602167].
- [28] G.W. Gibbons and N.S. Manton, “*The Moduli Space Metric for Well Separated BPS Monopoles*,” Phys. Lett. B **356** (1995) 32 [arXiv:hep-th/9506052].
- [29] E.J. Weinberg, “*Parameter Counting for Multimonopole Solutions*,” Phys. Rev. D **20** (1979) 936.
- [30] E.J. Weinberg, “*Fundamental Monopoles And Multi-Monopole Solutions For Arbitrary Simple Gauge Groups*,” Nucl. Phys. B **167** (1980) 500.
- [31] B. Julia and A. Zee, “*Poles with Both Magnetic and Electric Charges in Non-Abelian Gauge Theory*,” Phys. Rev. D **11** (1975) 2227.
- [32] N.S. Manton, “*The Force between 't Hooft-Polyakov Monopoles*,” Nucl. Phys. B **126** (1977) 525.
- [33] N.S. Manton, “*Monopole Interactions at Long Range*,” Phys. Lett. B **154** (1985) 397 [Erratum-ibid. **157B** (1985) 475].
- [34] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, (Princeton University Press, Princeton, 1988).
- [35] L. Alvarez-Gaume and D. Z. Freedman, “*Potentials For The Supersymmetric Nonlinear Sigma Model*,” Commun. Math. Phys. **91** (1983) 87.
- [36] D. Tong, “*A Note on $1/4$ -BPS States*,” Phys. Lett. B **460** (1999) 295 [arXiv:hep-th/9902005].
- [37] E.J. Weinberg and P. Yi, “*Magnetic Monopole Dynamics, Supersymmetry, and Duality*,” Phys. Rept. **438** (2007) 65-236. [hep-th/0609055].

- [38] G. 't Hooft, “*Computation of the Quantum Effects due to a Four-dimensional Pseudoparticle,*” Phys. Rev. D **14** (1976) 3432 [Erratum-ibid. D **18** (1978) 2199].
- [39] Y. Kazama, C.N. Yang and A.S. Goldhaber, “*Scattering Of A Dirac Particle With Charge Ze By A Fixed Magnetic Monopole,*” Phys. Rev. D **15** (1977) 2287.
- [40] T.T. Wu and C.N. Yang, “*Dirac Monopole without Strings: Monopole Harmonics,*” Nucl. Phys. B **107** (1976) 365.
- [41] H. Yamagishi, “*The Fermion Monopole System Reexamined,*” Phys. Rev. D **27** (1983) 2383-2396.
- [42] C. Callias, “*Index Theorems on Open Spaces,*” Commun. Math. Phys. **62** (1978) 213.
- [43] H.Y. Chen, N. Dorey and K. Petunin, “*Wall Crossing and Instantons in Compactified Gauge Theory,*” JHEP **1006** (2010) 024 [arXiv:1004.0703 [hep-th]].
- [44] D. Gaiotto, G.W. Moore and A. Neitzke, “*Four-dimensional Wall-crossing via Three-dimensional Field Theory,*” Commun. Math. Phys. **299** (2010) 163 [arXiv:0807.4723 [hep-th]].
- [45] D. Gaiotto, G.W. Moore and A. Neitzke, “*Wall-crossing, Hitchin Systems, and the WKB Approximation,*” arXiv:0907.3987 [hep-th].
- [46] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, “*Quantizing $N=2$ Multicenter Solutions,*” JHEP **0905** (2009) 002 [arXiv:0807.4556 [hep-th]].
- [47] J. Manschot, B. Pioline and A. Sen, “*Wall-Crossing from Boltzmann Black Hole Halos,*” arXiv:1011.1258 [hep-th].
- [48] D. Gaiotto, “*Surface Operators in $N=2$ 4d Gauge Theories,*” arXiv:0911.1316 [hep-th].
- [49] A. Hanany and K. Hori, “*Branes and $N=2$ Theories in Two-dimensions,*” Nucl. Phys. **B513** (1998) 119-174 [hep-th/9707192].
- [50] N. Dorey, “*The BPS Spectra of Two-dimensional Supersymmetric Gauge Theories with Twisted Mass Terms,*” JHEP **9811** (1998) 005 [hep-th/9806056].
- [51] N. Dorey, T.J. Hollowood, and D. Tong, “*The BPS Spectra of Gauge Theories in Two-dimensions and Four-dimensions,*” JHEP **9905** (1999) 006 [hep-th/9902134].
- [52] S. Lee and P. Yi, “*A Study of Wall-Crossing: Flavored Kinks in $D=2$ QED,*” JHEP **1003** (2010) 055 [arXiv:0911.4726 [hep-th]].